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From Types to Sets

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The aim is to construct the free topos generated by a category. Up to equivalence, one may assume that each topos is equipped with a canonical choice of representative subobjects satisfying certain obvious conditions, and one insists that morphisms between toposes preserve these exactly. Between the category of small categories and that of toposes one inserts the category of "dogmas" (Volger's closed logical categories). Dog is known to be equational over Cat, and it is here shown to be a reflective subcategory of Top. A *dogma* is a category with canonical finite products, it has a Heyting algebra object Ω which admits arbitrary objects as exponents such that, for each object A, the canonical morphism $\Omega \to \Omega^A$ has a left adjoint $\exists_A$ and a right adjoint $\forall_A$, and it satisfies the usual axiom of extensionality (interpreted in the obvious way). The construction of the topos generated by a dogma follows Volger: its objects are "sets" $1 \to \Omega^A$ and its morphisms are "relations" between sets which are universally defined and single valued. Inasmuch as a topos consists of sets, a dogma consists of types, and we find here much of traditional type theory in a categorical setting, which incorporates both Frege's process of set abstraction and something like Russell's theory of description.

INTRODUCTION

As is well-known, Frege's attempt to build mathematics on the theory of membership foundered on Russell's paradox. Russell himself came to the rescue with his theory of types. Apparently, this was rejected by most mathematicians as too cumbersome, even though Gödel used a simplified form of type theory in his famous 1931 paper on undecidable propositions and Church produced a very elegant formulation of type theory in 1940. Instead, mathematicians favor two rival methods for patching up the original theory, those of Gödel–Bernays and Zermelo–Fraenkel. Lawvere, in his elementary theory of the category of sets, rejected the theology of the membership relation altogether and replaced it by the dialectics of adjoint functors. Later Lawvere and Tierney introduced the now flourishing notion of an "elementary topos," which provides a bridge between algebraic geometry and logic. Their view is that the category of sets is only one of many toposes.

The present author became interested in an equational approach to foundations and, at the Halifax conference in 1971, proposed the name "dogma"

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for certain structured categories which were related to type theory à la Church as toposes are related to set theory:

$$\frac{\text{dogmas}}{\text{type theory}} = \frac{\text{toposes}}{\text{set theory}}.$$

Dogmas are "equational" over categories (or over graphs for that matter) in much the same way that groups are equational over sets, and one can easily construct the free dogma generated by any category.

Since toposes are dogmas, one may ask whether each dogma freely generates a topos. That this is indeed so seems to have occurred independently to several people (including the author), but priority clearly belongs to Volger, who presented his ideas in Oberwolfach in 1972 and widely distributed preprints of his [29], the publication of which was unduly delayed. Our dogmas are essentially his "closed logical categories"; in fact, both may be viewed as special cases of Lawvere's "hyperdoctrines." We have retained the term "dogma" for brevity and because the forgetful functor from Dog to Cat seems to amuse anglophone audiences.

Gadgets similar to dogmas have been investigated by Bénabou ("formal toposes") and Joyal. The essential point in Volger's construction of the free topos was rediscovered by Fourman, Coste, and Boileau, all of whom replaced Volger's closed logical categories by certain languages akin to type theory. (Similar languages had been investigated by many people, e.g., Mitchell, Freyd, Bénabou, Osius, and Rattray and Dana Schlomiuk.) The difference between their approach and ours may be expressed by yet another proportion:

$$\frac{\text{dogmas}}{\text{type theory}} = \frac{\text{algebraic theories à la Lawvere}}{\text{universal algebra à la Birkhoff}}$$

Aside from technical differences, there are three points in which the present elaboration differs from Volger's pioneering article. The first involves an approach through indeterminates already described in two earlier papers, really a categorical version of combinatory logic. The second is a detailed investigation of description in dogmas, a process which is known to work in toposes. The third point has to do with the universal property of the topos generated by a dogma: while Volger obtains the category of toposes as a full subcategory of Dog which is reflective up to isomorphism, we exhibit Top as a reflective subcategory of Dog which is full up to isomorphism.

Briefly, a *dogma* is a category with finite products, it has a Heyting algebra object Ω which admits powers Ω^A for all objects A, the canonical morphism $\Omega \to \Omega^A$ has a right adjoint $\forall_A$ and a left adjoint $\exists_A$, and the axiom of extensionality holds.

Given any object A of a dogma, one may adjoin an *indeterminate* morphism $x: 1 \to A$ to the dogma, much as one adjoins an indeterminate x to a commutative ring. Set abstraction is now provable (as a special case of λ-abstraction):

given any polynomial $p(x)\colon 1 \to \Omega$ (usually called a "propositional function"), there exists a unique morphism $f\colon A \to \Omega$ not involving x such that $fx = p(x)$. The associated morphism $\ulcorner f \urcorner\colon 1 \to \Omega^A$ is written $\{x \in A \mid p(x)\}$. Morphisms $1 \to \Omega^A$ are called "sets," although "predicates" might have been a better term. Of course, these sets are nonstandard, as long as we do not impose all the axioms the category of sets should satisfy. In particular, the underlying logic is intuitionistic.

Propositions appear as morphisms $1 \to \Omega$ in a dogma. Whenever the proposition $\forall_{x\in A}\, \exists!_{y\in B}\, p(x, y)$ is equal to the proposition *true*, one would like to exhibit a morphism $g\colon A \to B$ such that the proposition $\forall_{x\in A}\, p(x, gx)$ is equal to *true*. That this is possible in a Boolean topos was proved by Mitchell. We show that the same result holds in a dogma, provided only the singleton morphism $B \to \Omega^B$ is an equalizer of two morphisms into some power of Ω.

With Volger we assert that the topos $T(\mathcal{A})$ generated by a dogma $\mathcal{A}$ has as objects sets $1 \to \Omega^A$ and as morphisms functions, that is, binary relations between sets which are universally defined and single valued. One may ask whether the embedding $H\colon \mathcal{A} \to T(\mathcal{A})$ has the expected universal property: for every elementary topos $\mathscr{E}$ and every morphism $G\colon \mathcal{A} \to \mathscr{E}$ in the category of dogmas, there exists a unique morphism $G'\colon T(\mathcal{A}) \to \mathscr{E}$ in the category of toposes such that $G'H = G$. Such a result was obtained by Volger, but his G' was a pseudofunctor and it was only unique up to isomorphism. To assure that G' is a functor and unique, we assume that each object in a topos is provided with a canonical set of representative subobjects and that morphisms between toposes preserve these canonical subobjects. Fortunately, every elementary topos is equivalent to a topos with canonical subobjects.

The present results were presented at the Oberwolfach conference in 1974 and preliminary versions of this article were circulated in 1974, 1975, and 1977.

1. What Is x?

Concerning the nature of the variable x in mathematics, different points of view are possible. Physicists and writers of calculus textbooks are fond of regarding x as a variable quantity, while several people (Curry, Menger, Quine) have argued at times that variables could or should be done away with. We shall take the position that *variables are indeterminates*, as in algebra.

What is the situation in algebra? One adjoins an indeterminate x to a commutative ring A and obtains the so-called "polynomial ring" $A[x]$ together with a homomorphism $h\colon A \to A[x]$. This process has several aspects, all of which will be seen to give rise to significant analogies later.

(1) *Universal property*. For each ring homomorphism $f\colon A \to B$ and each element $b \in B$ there exists a unique ring homomorphism $f'\colon A[x] \to B$ such that $f'h = f$ and $f'(x) = b$.

(2) *Construction.* One defines a *polynomial* $f(x)$ as a word in the language of ring theory, e.g., $f(x) = (x + 2)\,x^2 + (-1)$, modulo an equivalence relation $=_x$ to assure the usual laws of ring theory, e.g., $(f(x) + g(x))\,h(x) =_x f(x)\,h(x) + g(x)\,h(x)$. $A[x]$ then consists of the polynomials (modulo $=_x$) with obvious ring operations, and $h\colon A \to A[x]$ sends the element a of A onto its equivalence class modulo $=_x$. One easily checks that the universal property is satisfied.

(3) *Normal form.* It appears that each polynomial can be written uniquely in the form $f(x) =_x a_0 + a_1 x + \cdots + a_n x^n$. Some writers like to define polynomials by such simplified expressions, but it should be noted that (3) is peculiar to commutative rings, while (1) and (2) hold equally for other algebraic systems, for example, for noncommutative rings.

(4) *Variables explained away.* In view of (3) above, one can define a polynomial as an essentially finite sequence of elements of A, but then the operations of addition and multiplication must be defined very carefully, the latter in a manner not intuitively obvious.

We are not interested in commutative rings here, but, for a start, in Cartesian categories, that is, categories with canonical finite products. For us a *Cartesian category* $\mathcal{A}$ will consist of two classes, the class $\mathcal{A}_0$ of objects (or *formulas*) and the class $\mathcal{A}_1$ of morphisms (or *proofs*), together with two mappings $\mathcal{A}_0 \to \mathcal{A}_1$ called "source" and "target." We write $f\colon A \to B$ to mean that the morphism f has the object A as source and the object B as target. (This may also be read: f is a "proof" of the "sequent" $A \to B$.) Among the objects there is a specified object 1, and for any two objects A and B there is a specified object $A \times B$. Moreover, we postulate the following *axioms*, *rules*, and *equations*:

Axioms:

$$1_A : A \to A \qquad 0_A : A \to 1 \qquad \pi_{A,B} : A \times B \to A$$

$$\pi'_{A,B} : A \times B \to B$$

Rules:

$$\frac{f\colon A \to B \;\; g\colon B \to C}{gf\colon A \to C} \qquad\qquad \frac{f\colon C \to A \;\; g\colon C \to B}{\langle f, g\rangle\colon C \to A \times B}$$

Equations:

$$f1_A = f \qquad\qquad f = 0_A \qquad\qquad \langle \pi_{A,B}h, \pi'_{A,B}h\rangle = h$$

$$1_B f = f \qquad\qquad \pi_{A,B}\langle f, g\rangle = f$$

$$(hg)f = h(gf) \qquad\qquad \pi'_{A,B}\langle f, g\rangle = g$$

(assuming $h\colon C \to D$) (assuming $f\colon A \to 1$) (assuming $h\colon C \to A \times B$).

Forgetting about the equations, we have here a deductive system, the "calculus of conjunction." It would be more familiar if we wrote $1 = T$, $A \times B = A \wedge B$.

Let A be an object of the Cartesian category $\mathcal{A}$, we shall attempt to adjoin an *indeterminate morphism* $x: 1 \to A$ to $\mathcal{A}$. We shall obtain the "polynomial cartesian category" $\mathcal{A}[x]$ together with a "Cartesian functor" $H: \mathcal{A} \to \mathcal{A}[x]$, that is, a functor which preserves the Cartesian structure exactly, e.g., $H(C \times B) = H(C) \times H(B)$, $H(\pi_{C,B}) = \pi_{H(C),H(B)}$. (Actually, $x: 1 \to H(A)$.)

(1) *Universal property.* For each Cartesian functor $F: \mathcal{A} \to \mathcal{B}$ and each morphism $b: 1 \to F(A)$ in $\mathcal{B}$ there exists a unique Cartesian functor $F': \mathcal{A}[x] \to \mathcal{B}$ such that $F'H = F$ and $F'(x) = b$.

(2) *Construction.* We form a "deductive system" as follows: Its formulas are those of $\mathcal{A}$, that is, the objects of $\mathcal{A}$. Its axioms are all morphisms $f: C \to B$ of $\mathcal{A}$, together with the "assumption" $x: 1 \to A$. Its rules are those of a Cartesian category. The set of proofs is the smallest set containing the axioms and closed under the rules. We then obtain a Cartesian category by introducing an equivalence relation $=_x$ between proofs. (It is understood that $f =_x g$ only if f and g have the same source and target.) In fact, $=_x$ is the smallest equivalence relation which respects the equations and implications between equations of a Cartesian category. For example, $(hg)f =_x h(gf)$ and, if $f =_x f'$ and $g =_x g'$, then $gf =_x g'f'$. To avoid cumbersome notation, we shall regard $=_x$ as the equality relation in $\mathcal{A}[x]$. One easily checks that the universal property (1) is satisfied.

(3) *Normal form.* Cartesian categories have the following property of *functional completeness*: Given any polynomial $f(x): C \to B$ in the indeterminate $x: 1 \to A$, there exists a unique morphism $g = \kappa_x f(x): A \times C \to B$ already in $\mathcal{A}$ such that $g\langle x0_C, 1_C\rangle =_x f(x)$. Forgetting about the equations, we extract from this a form of the *deduction theorem*: If $f(x): C \to B$ is a proof of the sequent $C \to B$ from the assumption $x: T \to A$, then there exists a proof of $A \wedge C \to B$ not depending on this assumption. We shall sketch a proof of functional completeness in Section 2.

(4) *Variables explained away.* It turns out that $A \times -: \mathcal{A} \to \mathcal{A}$ is the functor part of a cotriple on $\mathcal{A}$ and that functional completeness may be interpreted as saying that $\mathcal{A}[x]$ is the Kleisli category of this cotriple.

One may, of course, construct $\mathcal{A}[x]$ as this Kleisli category, with its curious definition of composition. However, the earlier construction (2) is more general, as it applies to other structured categories, e.g., to monoidal categories, for which functional completeness does not hold.

The observation that the Kleisli category of the above cotriple has the universal property of $\mathcal{A}[x]$ was already made by Volger [28], although he does not mention Kleisli categories by name. He calls our variables "constants," and he considers a set of constants at once.

2. Functional Completeness of Cartesian Categories

We shall now sketch a proof of functional completeness. The construction of $\kappa_x f(x)$ proceeds by induction on the length of $f(x)\colon C \to B$. We consider four cases: $f(x) = f$ does not depend on x; $f(x) = x$; $f(x) = \langle g(x), h(x)\rangle$; $f(x) = h(x)\, g(x)$. In the last two cases it is assumed that at least one of the $g(x)$ or $h(x)$ depends on x.

$$\kappa_x f = f\pi'_{A,C}\,,$$

$$\kappa_x x = \pi_{A,1}\,,$$

$$\kappa_x\langle g(x), h(x)\rangle = \langle \kappa_x g(x), \kappa_x h(x)\rangle,$$

$$\kappa_x(h(x)g(x)) = \kappa_x h(x)\,\langle \pi_{A,D}\,, \kappa_x g(x)\rangle,$$

where D is the source of $g(x)$.

One also checks by induction on the length of $f(x)$ that $\kappa_x f(x)\langle x0_C\,, 1_C\rangle =_x f(x)$. Finally, one must prove that $\kappa_x f(x)$ does not depend on the form of the polynomial $f(x)$: if $f(x) =_x g(x)$, then $\kappa_x f(x) = \kappa_x g(x)$. Indeed, the latter equation defines an equivalence relation $\equiv$ between polynomials which satisfies all the eight defining conditions of $=_x$. But $=_x$ was the smallest equivalence relation satisfying these eight conditions, hence $f(x) =_x g(x)$ implies $f(x) \equiv g(x)$.

For details of the above argument, the reader is referred to [16], where also the following three corollaries are drawn.

First, it is shown that $H_x\colon \mathcal{A} \to \mathcal{A}[x]$ is faithful, that is, for $f, g\colon C \to B$, $f =_x g$ implies $f = g$, provided $\mathrm{Hom}(1, A)$ is nonempty. That some such restriction is necessary is readily seen by considering a Cartesian category with an initial object $0 \neq 1$. Adjoining an indeterminate $x\colon 1 \to 0$, one easily deduces that $f =_x g$ in $\mathcal{A}[x]$, for any two morphisms $f, g\colon 1 \to B$ in $\mathcal{A}$. Indeed, if $0_0\colon 0 \to 1$, we have $f0_0 = g0_0$, hence $f0_0x =_x g0_0x$; but $0_0x =_x 1_1$, hence $f =_x g$. (See also Proposition 7.3 below.)

Second, the universal property (1) is exploited to extract from (3) the fact that, for any $a\colon 1 \to A$ in $\mathcal{A}$, $\kappa_x f(x)\langle a0_C\,, 1\rangle = f(a)$.

Third, we have the following special case of functional completeness, which is used frequently: Given any polynomial $f(x)\colon 1 \to B$ in the indeterminate $x\colon 1 \to A$, there exists a unique morphism $g\colon A \to B$ already in $\mathcal{A}$ such that $gx =_x f(x)$.

We shall investigate the effect of a Cartesian functor on $\kappa_x f(x)$. Let there be given a Cartesian functor $G\colon \mathcal{A} \to \mathcal{B}$ and indeterminates $x\colon 1 \to A$ over $\mathcal{A}$ and $x'\colon 1 \to G(A)$ over $\mathcal{B}$. In view of the universal property of $\mathcal{A}[x]$, we may extend G to a unique Cartesian functor $G'\colon \mathcal{A}[x] \to \mathcal{B}[x']$ such that $G'(x) = x'$ and the following square commutes:

$$\begin{tikzcd} \mathcal{A}[x] \arrow[r, "G'"] & \mathcal{B}[x'] \\ \mathcal{A} \arrow[u] \arrow[r, "G"'] & \mathcal{B} \arrow[u] \end{tikzcd}$$

Applying G' to the equation

$$\kappa_x f(x)\langle x 0_C, 1_C\rangle =_x f(x)$$

in $\mathcal{A}[x]$, we obtain

$$G(\kappa_x f(x))\langle x' 0_{G(C)}, 1_{G(C)}\rangle =_{x'} G'(f(x))$$

in $\mathcal{B}[x']$. It then follows from functional completeness that

$$G(\kappa_x f(x)) = \kappa_{x'} G'(f(x)).$$

We have thus shown the following result:

PROPOSITION 2.1. *Suppose $G: \mathcal{A} \to \mathcal{B}$ is a Cartesian functor and $x: 1 \to A$, $x': 1 \to G(A)$ are indeterminates over $\mathcal{A}$ and $\mathcal{B}$, respectively. If G is extended to $G': \mathcal{A}[x] \to \mathcal{B}[x']$ such that $G(x) = x'$, then $G(\kappa_x f(x)) = \kappa_{x'} G'(f(x))$.*

3. ONE INDETERMINATE SUFFICES

When it comes to two indeterminates, one has the choice of adjoining them simultaneously or consecutively. To simplify the discussion, we shall adopt the second point of view. Instead of writing $x: 1 \to A$, we shall often say that x is an *indeterminate of type A*.

Authors concerned with the language of type theory usually insist on listing a countable number of variables for each type, to be prepared for all eventualities. Indeed, if one wishes, one may adjoin a whole set or even class of indeterminates at once. However, in any particular application, a finite number of indeterminates will do since any formula that can be written down at all involves only a finite number of variables. As we shall see, already in a Cartesian category, a finite number of indeterminates may be replaced by a single one.

PROPOSITION 3.1. *If we successively adjoin indeterminates x of type A and y of type B to the Cartesian category $\mathcal{A}$, we obtain the Cartesian category $\mathcal{A}[x][y] \cong \mathcal{A}[z]$, where z is an indeterminate of type $A \times B$.*

Proof. Let $F: \mathcal{A}[z] \to \mathcal{A}[x][y]$ be the unique Cartesian functor such that $F(z) = \langle x, y \rangle$ and the following square commutes:

$$\begin{tikzcd} \mathcal{A}[z] \arrow[r, dashed, "F"] & \mathcal{A}[x][y] \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}[x] \arrow[u] \end{tikzcd}$$

Let $G_1: \mathcal{A}[x] \to \mathcal{A}[z]$ be the unique Cartesian functor such that $G_1(x) = \pi z$ and the following triangle commutes:

$$\begin{tikzcd} \mathcal{A}[x] \arrow[rr, dashed, "G_1"] & & \mathcal{A}[z] \\ & \mathcal{A} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Let $G: \mathcal{A}[x][y] \to \mathcal{A}[z]$ be the unique Cartesian functor such that $G(y) = \pi' z$ and the following triangle commutes:

$$\begin{tikzcd} \mathcal{A}[x][y] \arrow[rr, dashed, "G"] & & \mathcal{A}[z] \\ & \mathcal{A}[x] \arrow[ul] \arrow[ur, "G_1"'] & \end{tikzcd}$$

One easily checks that $GF(z) = z$ and that the following triangle commutes:

$$\begin{tikzcd} \mathcal{A}[z] \arrow[rr, "GF"] & & \mathcal{A}[z] \\ & \mathcal{A} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Therefore, GF is the identity functor on $\mathcal{A}[z]$.

Similarly, one shows that FG is the identity functor on $\mathcal{A}[x][y]$.

COROLLARY 3.2. *Every finite transcendental extension of a Cartesian category is isomorphic to a simple transcendental extension.*

We shall resist the temptation to discuss infinite transcendental extensions and refer the reader to Volger [28] instead.

In view of Proposition 3.1, we shall often denote an indeterminate of type $A \times B$ by $\langle x, y \rangle$, where x and y are indeterminates of types A and B, respectively.

4. The Category of Dogmas

Functional completeness holds not only for Cartesian categories, but also for Cartesian closed categories (for each B in $\mathcal{A}$, the functor $- \times B\colon \mathcal{A} \to \mathcal{A}$ is assumed to have a right adjoint), even for Cartesian closed categories with additional equational structure, as long as there are essentially no new rules (but new operations, axioms, and equations are permitted). Unfortunately, functional completeness does not hold for toposes, or even for categories with finite limits. Indeed, in that situation, that is, in the category of categories with finite limits, adjoining an indeterminate x of type A leads to $\mathcal{A}/A$ (see [11], 5.11.2), which is isomorphic to the Eilenberg–Moore category of the cotriple mentioned in (4) above.

Nonetheless, we may form $\mathcal{A}[x]$ inside the category of Cartesian categories, even when $\mathcal{A}$ is a topos, as long as we do not expect $\mathcal{A}[x]$ to be a topos. After all, when F is a field, $F[x]$ is a commutative ring and not a field, although there is a well-known process which embeds $F[x]$ into a field $F(x)$.

We shall see that, without loss of information, toposes may be approximated by certain systems that possess functional completeness. Such a system will be called a "dogma." Inasmuch as a topos is a categorical version of set theory, albeit nonstandard, a dogma is a categorical version of type theory. We shall see later that a dogma may be embedded into a topos in a manner analogous to the embedding of a commutative ring into a field.

A *predogma* is a Cartesian category $\mathcal{A}$ with a specified object Ω and for each object B of $\mathcal{A}$ an object PB of $\mathcal{A}$ satisfying the following new axiom, rule, and equations:

Axiom: $\epsilon_B : PB \times B \to \Omega.$

Rule: $$\frac{f\colon A \times B \to \Omega}{f^*\colon A \to PB}$$

Equations: $$\epsilon_B\langle f^*\pi_{A,B}\,,\, \pi'_{A,B}\rangle = f,$$

$$(\epsilon_B\langle g\pi_{A,B}\,,\, \pi'_{A,B}\rangle)^* = g.$$

We have thus assured a natural isomorphism $\mathrm{Hom}(A \times B, \Omega) \cong \mathrm{Hom}(A, PB)$ and we might have written Ω^B in place of PB.

In any predogma, one has

$$\mathrm{Hom}(A, \Omega) \cong \mathrm{Hom}(1 \times A, \Omega) \cong \mathrm{Hom}(1, PA).$$

It is often desirable to pass from a morphism $f\colon A \to \Omega$ to the associated "set" $\ulcorner f \urcorner\colon 1 \to PA$ (the corners are Lawvere's, not Quine's):

$$\ulcorner f \urcorner = (f\pi'_{1\,A})^*.$$

In the converse direction, with any "set" $\alpha: 1 \to PA$ one associates the morphism $\alpha^{(}: A \to \Omega$, where

$$\alpha^{(} = \epsilon_A \langle \alpha 0_A , 1_A \rangle.$$

These two constructions are illustrated by the following "proof trees":

$$\dfrac{\dfrac{1 \times A \to A \qquad A \xrightarrow{f} \Omega}{1 \times A \to \Omega}}{1 \to PA} \qquad \dfrac{\dfrac{\dfrac{A \to 1 \qquad 1 \xrightarrow{\alpha} PA}{A \to PA} \quad A \to A}{A \to PA \times A} \qquad PA \times A \to \Omega}{A \to \Omega}$$

A *dogma* is a predogma with additional axioms (morphisms):

$$\begin{aligned} &true, false\colon 1 \to \Omega, \\ &\wedge, \vee, \Rightarrow\colon \Omega \times \Omega \to \Omega, \\ &\forall_A , \exists_A\colon PA \to \Omega, \end{aligned}$$

which satisfy an essentially finite number of equations.

Instead of writing these equations down, we shall first say what we want to accomplish with them. We want to use the above data to define a binary relation $\leqslant_A$ (usually written without the subscript) on each $\mathrm{Hom}(A, \Omega)$. This relation is to satisfy the following conditions (we write $f \wedge g$ for $\wedge\langle f, g\rangle$, etc.):

(1) $f \leqslant f$;

(2) if $f \leqslant g$ and $g \leqslant f$, then $f = g$;

(3) if $f \leqslant g$ and $g \leqslant h$, then $f \leqslant h$;

(4) if $f \leqslant g$, then $fk \leqslant gk$;

(5) $f \leqslant true\ 0_A$,

(5′) $false\ 0_A \leqslant f$,

(6) $h \leqslant f \wedge g$ iff $h \leqslant f$ and $h \leqslant g$;

(6′) $f \vee g \leqslant h$ iff $f \leqslant h$ and $g \leqslant h$;

(7) $f \wedge g \leqslant h$ iff $g \leqslant f \Rightarrow h$;

(8) $f \leqslant \forall_B \varphi^*$ iff $f\pi_{A,B} \leqslant \varphi$;

(8′) $\exists_B \varphi^* \leqslant f$ iff $\varphi \leqslant f\pi_{A,B}$; for all $f, g, h\colon A \to \Omega$, $k\colon B \to A$ and $\varphi\colon A \times B \to \Omega$;

(9) the axiom of extensionality (we defer a statement of this until Section 6).

What do these conditions mean?

Conditions (1) to (4) assert that Ω is an *ordered* object of the category $\mathcal{A}$, in the following sense: the functor $\mathrm{Hom}(-, \Omega)$: $\mathcal{A}^{\mathrm{op}} \to$ Sets can be lifted to the category Ord of partially ordered sets, that is, there exists a functor $\mathcal{A}^{\mathrm{op}} \to$ Ord such that the following triangle commutes:

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathrm{Hom}(-,\Omega)"'] & & \mathrm{Ord} \arrow[dl] \\ & \mathrm{Sets} & \end{tikzcd}$$

where Ord $\to$ Sets is the usual forgetful functor.

One easily sees that finite products and internal powers of ordered objects (if they exist) are also ordered. In particular, since Ω is ordered so are 1, $\Omega \times \Omega$, and PB. Indeed, the order of $\mathrm{Hom}(A, 1)$ is trivial, the order of $\mathrm{Hom}(A, \Omega \times \Omega)$ is defined by

$$\langle f, g\rangle \leqslant \langle f', g'\rangle \quad \text{iff} \quad f \leqslant f' \quad \text{and} \quad g \leqslant g',$$

and the order of $\mathrm{Hom}(A, PB)$ is defined by comparison with $\mathrm{Hom}(A \times B, \Omega)$, that is, for φ, ψ: $A \to PB$,

$$\varphi \leqslant \psi \quad \text{iff} \quad \varphi^{+} \leqslant \psi^{+},$$

where $\varphi^{+} = \epsilon_B\langle \varphi\pi_{A,B}, \pi'_{A,B}\rangle$.

If Ω_1 and Ω_2 are ordered objects, it becomes possible to say that u: $\Omega_1 \to \Omega_2$ has a *left adjoint* v: $\Omega_2 \to \Omega_1$, namely,

$$f_2 \leqslant uf_1 \quad \text{iff} \quad vf_2 \leqslant f_1$$

for all f_1: $A \to \Omega_1$ and f_2: $B \to \Omega_2$. A more concise way of saying this is

$$1_{\Omega_2} \leqslant uv, \qquad vu \leqslant 1_{\Omega_1}.$$

Conditions (5), (6), and (8) and (5′), (6′), and (8′) now assert that

$$0_\Omega : \Omega \to 1, \qquad \langle 1_\Omega, 1_\Omega\rangle: \Omega \to \Omega \times \Omega, \qquad \pi^{*}_{\Omega,A} : \Omega \to PA$$

have right adjoints

$$\mathit{true}, \qquad \wedge, \qquad \forall_A$$

and left adjoints

$$\mathit{false}, \qquad \vee, \qquad \exists_A,$$

respectively. Condition (7) also has the form of an adjunction.

It is clear from the above conditions that

$$f \leqslant g \qquad \text{iff} \quad f \wedge g = f,$$

so this could serve as a definition of $\leqslant$. Another possibility would have been: $f \Rightarrow g = \mathit{true}$.

Conditions (1) to (4) may now be replaced by three equations:

$$f \wedge f = f, \qquad f \wedge g = g \wedge f, \qquad (f \wedge g) \wedge h = f \wedge (g \wedge h),$$

which are to hold for all f, g, and h.

It is not difficult to restate the equations without the implicit quantification over elements of $\mathrm{Hom}(A, \Omega)$. In fact, taking $A = \Omega$ and $f = 1_\Omega$, the first equation specializes to

$$\wedge \langle 1_\Omega, 1_\Omega \rangle = 1_\Omega,$$

and multiplying this equation by f on the right, one recaptures $f \wedge f = f$. In the same way, the second equation may be rewritten

$$\wedge = \wedge \langle \pi'_{A,B}, \pi_{A,B} \rangle,$$

and a similar reformulation applies to the third equation.

The adjointness relations (5), (6), and (8) may be written as follows:

$$1_\Omega \leqslant \mathit{true}\ 0_\Omega,$$

$$1_\Omega \leqslant \wedge \langle 1_\Omega, 1_\Omega \rangle, \qquad \wedge \leqslant \pi_{\Omega,\Omega}, \qquad \wedge \leqslant \pi'_{\Omega,\Omega},$$

$$1_\Omega \leqslant \forall_A \overset{*}{\pi}_{\Omega,A}, \qquad \forall_A \pi_{PA,A} \leqslant \epsilon_A$$

as is easily seen. Their duals (5′), (6′), and (8′) may be reformulated similarly, and (7) may be written

$$\pi'_{\Omega,\Omega} \leqslant \Rightarrow \langle \pi_{\Omega,\Omega}, \wedge \rangle, \qquad \wedge \langle \pi_{\Omega,\Omega}, \Rightarrow \rangle \leqslant \pi'_{\Omega,\Omega}.$$

These 14 inequalities (depending on the object A) may be replaced by 14 equations if $\leqslant$ is eliminated in favor of $\wedge$ and $=$.

Together with the axiom of extensionality, the discussion of which has been postponed, we thus have 18 equations (depending on A). If it were desirable, we could combine these 18 equations into one, by repeated use of the rule that

$$f \leqslant f' \quad \text{and} \quad g \leqslant g' \qquad \text{iff} \quad \langle f, g \rangle \leqslant \langle f', g' \rangle.$$

As morphisms between dogmas (predogmas) we shall take functors which preserve their structure exactly. We shall call such functors *orthodox* (*pre-*

orthodox). Clearly, a preorthodox functor is Cartesian. Moreover, a preorthodox functor is orthodox if and only if it preserves *true*, *false*, $\wedge$, $\vee$, $\Rightarrow$, $\forall$, and $\exists$. We shall see later that it suffices for some of these morphisms to be preserved. (To say that $G: \mathcal{A} \to \mathcal{B}$ preserves $\forall$ means that $G(\forall_A) = \forall_{G(A)}$.)

5. Functional Completeness for Dogmas

For esthetic and other reasons that will become clear later, we shall from now on replace the symbol $=$ denoting equality in a dogma by $\cdot{=}\cdot$.

Given an object A of the dogma $\mathcal{A}$, one may adjoin an indeterminate $x: 1 \to A$ as in Section 1, only now there are many additional axioms and equations and one additional rule *, which affect the definition of "proof." We again take $\cdot{=}\cdot_x$ as the smallest equivalence relation on proofs from the assumption $x: 1 \to A$ which respects all equations and implications between equations.

Theorem 5.1. *Functional completeness holds for predogmas and dogmas.*

Proof. We shall modify the proof sketched in Section 2 for Cartesian categories. One additional case now arises in the recursive definition of $\kappa_x f(x)$.

Suppose $f(x): C \times B \to \Omega$, then $f(x)^*: C \to PB$. We put

$$\kappa_x(f(x)^*) \cdot{=}\cdot (\kappa_x f(x)\, \alpha_{A,C,B})^*,$$

where $\alpha_{A,C,B}: (A \times C) \times B \to A \times (C \times B)$ is the associativity morphism defined by

$$\alpha_{A,C,B} \cdot{=}\cdot \langle \pi_{A,C}\pi_{A\times C,B}\,, \langle \pi'_{A,C}\pi_{A\times C,B}\,, \pi'_{A\times C,B}\rangle\rangle.$$

The rest of the proof proceeds as above. See Proposition 3.5 of [16] for the details.

We introduce some useful terminology: morphisms $1 \to A$ are called *entities of type A*; in particular, entities of type Ω are *propositions* and entities of type PA are *sets*.

Corollary 5.2. *Given an indeterminate x of type A in a dogma and a polynomial $p(x)$ of type Ω (propositional function), there is a unique set $\alpha \cdot{=}\cdot \{x \mid p(x)\}$ of type PA such that*

$$\epsilon_A\langle \alpha, x\rangle \cdot{=}\cdot_x p(x).$$

Proof. Take

$$\alpha \cdot{=}\cdot ((\kappa_x f(x))\, \gamma_{1,A})^*,$$

where $\gamma_{B,A}: B \times A \to A \times B$ is the commutativity morphism defined by

$$\gamma_{B,A} \cdot\!=\!\cdot \langle \pi_{B,A}, \pi_{B,A} \rangle.$$

This corollary may be viewed as a categorical version of *set abstraction*. This becomes even more evident if we rewrite $\epsilon_A\langle \alpha, x \rangle$ as $x \in \alpha$.

As at the end of Section 2, we also have the following:

COROLLARY 5.3. *For any entity a of type A and any propositional function p(x) in the indeterminate x of type A in a dogma*

$$a \in \{x \mid p(x)\} \cdot\!=\!\cdot p(a).$$

It is sometimes useful to indicate the type of an indeterminate explicitly. Thus we may replace x by $x \in A$ and write

$$a \in \{x \in A \mid p(x)\} \cdot\!=\!\cdot p(a).$$

Let us also note the abbreviation

$$\{\langle x, y \rangle \in A \times B \mid p(x, y)\} \cdot\!=\!\cdot \{z \in A \times B \mid p(\pi_{A,B} z, \pi'_{A,B} z)\}.$$

In view of functional completeness, Proposition 2.1 remains valid in a dogma or predogma. In particular, we have the following application:

COROLLARY 5.4. *Suppose $G: \mathcal{A} \to \mathcal{B}$ is a (pre)orthodox functor and $x: 1 \to A$ and $x': 1 \to G(A)$ are indeterminates over $\mathcal{A}$ and $\mathcal{B}$, respectively. If G is extended to G′ such that $G'(x) = x'$, then*

$$G(\{x \in A \mid p(x)\}) \cdot\!=\!\cdot \{x' \in G(A) \mid G'(p(x'))\}.$$

Of course there is no harm in replacing x' by x.

6. INTERNAL LANGUAGE OF A DOGMA

By the *internal language* of a dogma $\mathcal{A}$ or $\mathcal{A}[x]$ we mean the set of all propositions, that is, morphisms $1 \to \Omega$. This internal language must be distinguished from the external or metalanguage, which is the theory of dogmas with names for the morphisms of $\mathcal{A}$.

If we do not distinguish between propositions and the expressions which denote them, we may regard the internal language as part of the metalanguage. Sometimes (as in Proposition 12.1 below) it may be possible to recapture the external language (or at least its equational part) from the internal one, but in general this need not be the case.

We shall not spell out, in the manner dear to logicians, recursive definitions of terms, formulas, etc., for the internal language. For us, this is merely the set of propositions, but among these are all the usual formulas of typed predicate calculus, as will be seen.

First of all, we have the following "atomic" propositions:

$$\textit{true}, \textit{false}, a \in \alpha \qquad (\text{meaning } \epsilon_A\langle \alpha, a\rangle).$$

If p and q are propositions, we write

$$\begin{aligned}
&\neg p \text{ for } \Rightarrow\langle p, \textit{false}\rangle,\\
&p \wedge q \text{ for } \bigwedge\langle p, q\rangle,\\
&p \vee q \text{ for } \bigvee\langle p, q\rangle,\\
&p \Rightarrow q \text{ for } \Rightarrow\langle p, q\rangle,\\
&p \Leftrightarrow q \text{ for } (p \Rightarrow q) \wedge (q \Rightarrow p).
\end{aligned}$$

If $p(x)$ is a propositional function in the indeterminate $x \in A$, we write

$$\begin{aligned}
&\forall_{x\in A}\, p(x) \text{ for } \forall_A\{x \in A \mid p(x)\},\\
&\exists_{x\in A}\, p(x) \text{ for } \exists_A\{x \in A \mid p(x)\}.
\end{aligned}$$

If α and β are sets of type PA, we write

$$\alpha \subseteq \beta \text{ for } \forall_{x\in A}(x \in \alpha \Rightarrow x \in \beta).$$

The symbol $\subseteq$ is not to be confused with the symbol $\leqslant$, which is part of the metalanguage. Incidentally, the $\in$ in "$x \in A$" belongs to the metalanguage too.

If a and b are of type A, we write

$$a = b \text{ for } \forall_{y\in PA}(a \in y \Leftrightarrow b \in y),$$

following Leibniz. The symbol $=$ should not be confused with the equality symbol in the metalanguage; this is the main reason why from Section 5 on we have been writing the latter as $\cdot = \cdot$.

If a is an entity of type A, we write

$$\{a\} \text{ for } \{x \in A \mid x = a\}.$$

We also write

$$\exists!_{x\in A}\, p(x) \text{ for } \exists_{x'\in A}(\{x \in A \mid p(x)\} = \{x'\}).$$

In the presence of extensionality [see (9) below] this may be rewritten as

$$\exists_{x\in A}\, \forall_{x'\in A}(p(x') \Leftrightarrow x' = x).$$

The symbol $\leqslant$ is a symbol of the metalanguage as has already been mentioned. To bring us closer to traditional logic, we shall write

$$p \vdash q \text{ for } p \leqslant q, \text{ that is, for } p \wedge q \;\cdot\!=\!\cdot\; p,$$

when p and q are propositions; similarly

$$p(x) \vdash_x q(x) \text{ for } p(x) \wedge q(x) \;\cdot\!=\!\cdot_x\; p(x),$$

where $p(x)$ and $q(x)$ are propositional functions.

Moreover, we write

$$\vdash q \text{ for } \mathit{true} \vdash q, \text{ that is, for } \mathit{true} \;\cdot\!=\!\cdot\; q.$$

The symbol $\vdash_x$ may be used to cast the definition of a dogma into a form more familiar to logicians.

We recall that, in view of functional completeness, there is a one-to-one correspondence between morphisms $f\colon A \to \Omega$ in the predogma $\mathcal{O}\!\ell$ and propositional functions $p(x)\colon 1 \to \Omega$ in $\mathcal{O}\!\ell[x]$, where x is an indeterminate of type A. This may be utilized to replace the inequality $f \leqslant g$ by the more familiar entailment $p(x) \vdash_x q(x)$. An indeterminate of type 1 is usually omitted, so that $p \vdash q$ may also be regarded as a special case of this.

Thus, a dogma may be described as a predogma which satisfies the following conditions:

(1) $p(x) \vdash_x p(x)$;

(2) if $p(x) \vdash_x q(x)$ and $q(x) \vdash_x p(x)$, then $p(x) \;\cdot\!=\!\cdot_x\; q(x)$;

(3) if $p(x) \vdash_x q(x)$ and $q(x) \vdash_x r(x)$, then $p(x) \vdash_x r(x)$;

(4) if $p(x) \vdash_x q(x)$, then $p(g(y)) \vdash_y q(g(y))$;

(5) $p(x) \vdash_x \mathit{true}$;

(5′) $\mathit{false} \vdash_x p(x)$;

(6) $r(x) \vdash_x p(x) \wedge q(x)$ iff $r(x) \vdash_x p(x)$ and $r(x) \vdash_x q(x)$;

(6′) $p(x) \vee q(x) \vdash_x r(x)$ iff $p(x) \vdash_x r(x)$ and $q(x) \vdash_x r(x)$;

(7) $p(x) \wedge q(x) \vdash_x r(x)$ iff $q(x) \vdash_x p(x) \Rightarrow r(x)$;

(8) $p(x) \vdash_x \forall_{y\in B} s(x, y)$ iff $p(x) \vdash_{\langle x,y\rangle} s(x, y)$;

(8′) $\exists_{y\in B} s(x, y) \vdash_x p(x)$ iff $s(x, y) \vdash_{\langle x,y\rangle} p(x)$, for all $p(x)$, $q(x)$, $r(x)$: $1 \to \Omega$, $g(y)$: $1 \to A$ and $s(x, y)$: $1 \to \Omega$, where x and y are indeterminates of types A and B, respectively. Finally we state the axiom of extensionality, which has been deferred until now:

(9) $\quad \vdash \forall_{u \in PA} \forall_{v \in PA}(\forall_{x \in A}(x \in u \Leftrightarrow x \in v) \Rightarrow (u = v))$.

We note the following special case of extensionality, which is obtained from (9) by taking $A = 1$ and noting that $P1 \cong \Omega$:

(9′) $\quad \vdash \forall_{u \in \Omega} \forall_{v \in \Omega}((u \Leftrightarrow v) \Rightarrow (u = v))$.

Since many of the results that follow do not require extensionality, we shall indicate its use explicitly in cases where it is needed.

7. More about Deduction in a Dogma

Not surprisingly, the symbol $\vdash$ obeys the usual rules of intuitionistic natural deduction, some of which appear in Section 6. We also have the deduction theorem:

$$p \vdash q \qquad \text{if and only if} \quad \vdash p \Rightarrow q$$

as well as the usual rules of "universal specification" and "existential generalization": For any entity a of type A,

$$\forall_{x \in A}\, q(x) \vdash q(a), \qquad q(a) \vdash \exists_{x \in A}\, q(x).$$

For example, the first of these is shown as follows. Clearly,

$$\forall_{x \in A}\, q(x) \vdash \forall_{x \in A}\, q(x).$$

Therefore, by (8) above (with x of type 1 omitted and y replaced by x), we have

$$\forall_{x \in A}\, q(x) \vdash_x q(x).$$

In view of Corollary 5.4, we may apply the "substitution functor" (the unique orthodox functor F: $\mathcal{O}[x] \to \mathcal{O}$ extending the identity functor $\mathcal{O} \to \mathcal{O}$ such that $F(x) = a$) to this and obtain

$$\forall_{x \in A}\, q(x) \vdash q(a).$$

We also note the following useful equations:

$$\forall_{x \in A} \forall_{y \in B}\, p(x, y) \;\cdot\!\!=\!\!\cdot\; \forall_{\langle x, y \rangle \in A \times B}\, p(x, y),$$
$$\exists_{x \in A} \exists_{y \in B}\, p(x, y) \;\cdot\!\!=\!\!\cdot\; \exists_{\langle x, y \rangle \in A \times B}\, p(x, y).$$

For example, in the first equation we go from left to right by two universal specifications followed by one universal generalization (half of (8)) and from right to left by one universal specification followed by two universal generalizations.

We also have the following familiar rules of equality:

$$\vdash a = a; \qquad a = b \vdash b = a; \qquad a = b \wedge b = c \vdash a = c;$$
$$a = b \vdash f(a) = f(b); \qquad p = q \vdash p \Leftrightarrow q.$$

Here a, b, and c are entities of type A, $f(x)$ is a polynomial of type B in an indeterminate of type A, and p and q are propositions.

For example, we shall prove the last two of these. By definition, of equality,

$$a = b \vdash \forall_{y \in PA}(a \in y \Leftrightarrow b \in y).$$

Hence, by universal specification,

$$a = b \vdash a \in \alpha \Leftrightarrow b \in \alpha,$$

where α is any set of type PA.

In particular, take $\alpha \cdot\!=\!\cdot \{x \in A \mid f(a) = f(x)\}$. Then $a \in \alpha \cdot\!=\!\cdot f(a) = f(a)$ and $b \in \alpha \cdot\!=\!\cdot f(a) = f(b)$, hence $a = b \vdash f(a) = f(a) \Leftrightarrow f(a) = f(b)$. Since $\vdash f(a) = f(a)$, it follows that $a = b \vdash f(a) = f(b)$.

On the other hand, take $a \cdot\!=\!\cdot p$, $b \cdot\!=\!\cdot q$, and $\alpha \cdot\!=\!\cdot \{t \in \Omega \mid t\}$, then $a \in \alpha \cdot\!=\!\cdot p$ and $b \in \alpha \cdot\!=\!\cdot q$, hence $p = q \vdash p \Leftrightarrow q$.

We shall make occasional use of the following:

$$\exists_{x \in A}(p(x) \wedge x = a) \cdot\!=\!\cdot p(a),$$

which is easily proved.

Of interest is the following connection between the equality in the metalanguage and the equality in a dogma.

PROPOSITION 7.1. *If $f, g\colon A \to B$ are morphisms in a dogma, then $f \cdot\!=\!\cdot g$ implies $\vdash \forall_{x \in A} fx = gx$. The converse holds if $B = \Omega$ or $B = PC$.*

Proof. The direct implication is evident. In the converse direction, first assume $B = \Omega$.

Suppose $\vdash \forall_{x \in A} fx = gx$, then $\vdash_x fx = gx$. But $fx = gx \vdash_x fx \Leftrightarrow gx$, hence $\vdash_x fx \Leftrightarrow gx$. From this we infer $fx \vdash_x gx$ and $gx \vdash_x fx$, hence $fx \cdot\!=\!\cdot_x gx$. Thus, by functional completeness, $f \cdot\!=\!\cdot g$.

Next, assume $B = PC$. As above, we deduce that $\vdash_{\langle x, y\rangle} y \in fx \Leftrightarrow y \in gx$. Let $f^+, g^+ : A \times C \to \Omega$ correspond to f, g, then $\vdash_z f^+z \Leftrightarrow g^+z$. From this we deduce $f^+ \cdot\!=\!\cdot g^+$ as above, hence $f \cdot\!=\!\cdot g$.

COROLLARY 7.2. *If a and b are entities of type A in a dogma, then* $\{a\} \cdot\!=\!\cdot \{b\}$ *if and only if* $\vdash a = b$. *When* $A = \Omega$ *or* $A = PC$, *this implies* $a \cdot\!=\!\cdot b$.

Proof. Suppose $\{a\} \cdot\!=\!\cdot \{b\}$, then $\vdash a \in \{a\} \Leftrightarrow a \in \{b\}$. But $a \in \{a\} \cdot\!=\!\cdot$ *true*, hence $\vdash a \in \{b\}$, that is, $\vdash a = b$.

Conversely, suppose $\vdash a = b$, then $x = a \cdot\!=\!\cdot_x x = b$, that is, $\{a\} \text{`}x \cdot\!=\!\cdot_x \{b\} \text{`}x$, and so $\{a\} \cdot\!=\!\cdot \{b\}$, by functional completeness.

Now assume $A = PC$ ($\Omega \cong P1$ is a special case of this). From $\vdash a = b$ we deduce $\vdash_x \forall_{y \in 1} (x \in a) y = (x \in b) y$. Applying Proposition 7.1 to the dogma $\mathfrak{A}[x]$, we obtain $x \in a \cdot\!=\!\cdot_x x \in b$, from which $a \cdot\!=\!\cdot b$ by functional completeness.

PROPOSITION 7.3. *If x is an indeterminate of type A in a dogma* $\mathfrak{A}$, *then* $H_x: \mathfrak{A} \to \mathfrak{A}[x]$ *is faithful if and only if* $\pi'_{A,C}: A \times C \to C$ *is epi for all objects C. This is true, in particular, if* Hom(1, A) *is nonempty.*

Proof. H_x is faithful if and only if, for all $f, g: C \to B$, $f \cdot\!=\!\cdot_x g$ implies $f \cdot\!=\!\cdot g$. In view of functional completeness, $f \cdot\!=\!\cdot_x g$ if and only if $\kappa_x f \cdot\!=\!\cdot \kappa_x g$, that is, $f\pi'_{A,C} \cdot\!=\!\cdot g\pi'_{A,C}$. This will imply that $f \cdot\!=\!\cdot g$ if and only if $\pi'_{A,C}$ is epi.

When $a: 1 \to A$, we may infer that $f \cdot\!=\!\cdot f\pi'_{A,C}\langle a, 1_C\rangle \cdot\!=\!\cdot g\pi'_{A,C}\langle a, 1_C\rangle \cdot\!=\!\cdot g$.

COROLLARY 7.4. *If* Hom(1, A) *is nonempty, then from* $p \vdash_x q$ *one may infer* $p \vdash q$.

To illustrate the significance of the above restriction on A, observe that in any dogma

$$\text{(i)} \quad \forall_{x \in A}\, p(x) \vdash_x \exists_{x \in A}\, p(x),$$

as we may deduce this from

$$\forall_{x \in A}\, p(x) \vdash_x p(x)$$

and

$$p(x) \vdash_x \exists_{x \in A}\, p(x).$$

However, we must not infer

$$\text{(ii)} \quad \forall_{x \in A}\, p(x) \vdash \exists_{x \in A}\, p(x),$$

which may well be false, for example, in the dogma of sets if A is the empty set.

On the other hand, we may demonstrate (ii) when there is an entity a of type A. This may be seen, even without using Corollary 7.4, by observing that

$$\forall_{x \in A}\, p(x) \vdash p(a)$$

and

$$p(a) \vdash \exists_{x \in A}\, p(x).$$

Note that Proposition 7.3 (but not Corollary 7.4) already holds in a Cartesian category.

We shall now discuss a characterization of orthodox functors. But first we make the observation that in type theory one may quantify over propositional variables.

Not only do we have

$$\mathit{false} \vdash p$$

for any proposition p, but also

$$\mathit{false} \vdash_w w$$

for an indeterminate w of type Ω. From this we easily deduce

(a) $\vdash \forall_{w\in\Omega}\, \mathit{false} \Rightarrow w.$

In the same manner, one establishes

(b) $\vdash \forall_{u\in\Omega} \forall_{v\in\Omega} \forall_{w\in\Omega} ((u \vee v) \Rightarrow w) \Leftrightarrow ((u \Rightarrow w) \wedge (v \Rightarrow w)))$

and

(c) $\vdash \forall_{y\in PA} \forall_{w\in\Omega} ((\exists_{x\in A}\, y^{\prime}x \Rightarrow w) \Leftrightarrow \forall_{x\in A}\, (y^{\prime}x \Rightarrow w)).$

Remark 7.5. A preorthodox functor $G: \mathcal{A} \to \mathcal{B}$ between dogmas is orthodox if and only if it preserves *true*, $\wedge$, $\Rightarrow$, and $\forall$.

Proof. Suppose the preorthodox functor G does preserve *true*, $\wedge$, $\Rightarrow$, and $\forall$. We shall show, for example, that it preserves $\vee$. In view of Corollary 5.4, we may apply G to (b) and obtain

(b′) $\vdash \forall_{u\in\Omega} \forall_{v\in\Omega} \forall_{t\in\Omega}\ ((u \vee' v) \Rightarrow t) \Leftrightarrow ((u \Rightarrow t) \wedge (v \Rightarrow t))$ in $\mathcal{B}$, where $\vee' \cdot\!=\!\cdot\, G(\vee)$. But (b) also holds in $\mathcal{B}$, and from (b) and (b′) we easily obtain

$$\vdash \forall_{u\in\Omega} \forall_{v\in\Omega} ((u \vee' v) \Leftrightarrow (u \vee v)),$$

from which one may deduce that $\vee' \cdot\!=\!\cdot \vee$.

In the same way, letting $G(\exists_A) \cdot\!=\!\cdot\, \exists'_{G(A)}$, we deduce from (c) that

$$\vdash \forall_{y\in PG(A)} (\exists'_{G(A)} y \Leftrightarrow \exists_{G(A)} y),$$

from which it follows that $\exists'_{G(A)} \cdot\!=\!\cdot\, \exists_{G(A)}$, by Proposition 7.1.

Remark 7.6. As Prawitz has already pointed out, *false*, $\vee$, *and* $\exists$ may actually be defined in terms of *true*, $\wedge$, $\Rightarrow$, and $\forall$. Thus we may put

$$\mathit{false} \cdot\!=\!\cdot\, \forall_{t\in\Omega}\, t,$$

$$p \vee q \cdot\!=\!\cdot\, \forall_{t\in\Omega} (((p \Rightarrow t) \wedge (q \Rightarrow t)) \Rightarrow t),$$

$$\exists_{x\in A}\, p(x) \cdot\!=\!\cdot\, \forall_{t\in\Omega} (\forall_{x\in A}\, (p(x) \Rightarrow t) \Rightarrow t),$$

and deduce conditions (5′), (6′), and (8′) of Section 6.

This observation implies that the definition of a dogma given in Section 6 may be simplified by omitting conditions (5′), (6′), and (8′).

Remark 7.7. In a dogma (with extensionality) we may define $\wedge$, $\Rightarrow$, and $\forall$ in terms of $=$. Thus a preorthodox functor between dogmas (with extensionality) is orthodox if and only if preserves *true* and $=$. Indeed

$$\begin{aligned} p \wedge q &\cdot\!=\!\cdot \langle p, q\rangle = \langle \mathit{true}, \mathit{true}\rangle, \\ p \Rightarrow q &\cdot\!=\!\cdot (p \wedge q) = p, \\ \forall_{x\in A}\, p(x) &\cdot\!=\!\cdot \{x \in A \mid p(x)\} = \{x \in A \mid \mathit{true}\}. \end{aligned}$$

Proof. For example, the right-hand side of the last equation may be rewritten with the help of (9) as $\forall_{x\in A}\,(p(x) \Leftrightarrow \mathit{true})$, that is, $\forall_{x\in A}\, p(x)$.

Note that we could also write

$$\mathit{true} \cdot\!=\!\cdot 0_1 = 0_1 ,$$

but we shall have no need to do so.

8. Description in a Dogma

Ever since Bertrand Russell, logicians are fond of postulating a description operator, which, supposing that $\vdash\!\exists!_{y\in B}\, p(y)$, allows them to name an entity b such that $\vdash\! p(b)$. Mitchell first observed that such an entity b exists in a Boolean topos, even if $p(y)$ involves another variable. We shall see that this result holds already in a dogma, provided the "singleton morphism" $\iota_B\colon B \to PB$ is an equalizer of two morphisms into some PC.

The *singleton* morphism ι_B is most easily defined by functional completeness as the unique morphism $B \to PB$ such that $\iota_B y \cdot\!=\!\cdot_y \{y\}$, for an indeterminate y of type B. It may also be described as $\iota_B \cdot\!=\!\cdot \delta_B^*$, where $\ulcorner\delta_B\urcorner \cdot\!=\!\cdot \{\langle y, y'\rangle \in B \times B \mid y = y'\}$, as is easily verified.

A morphism $m\colon B \to A$ will be called a *P-regular monomorphism* if it is an equalizer of two morphisms $f, g\colon A \to PC$ for some object C. We shall see presently that $\iota_B\colon B \to PB$ is a P-regular monomorphism if $B = \Omega$ or $B = PC$. In general, there is no reason to assume that ι_B is even a monomorphism. When $B = \Omega$ or $B = PC$ this is easily deduced from Corollary 7.2 and functional completeness.

Lemma 8.1. *Let m be a P-regular monomorphism in a dogma, and suppose that $\vdash\! \forall_{x\in A}\, \exists_{y\in B}\, my = fx$, where $f\colon A \to C$. Then there exists a (unique) g such that $mg = f$.*

Proof. Let m be the equalizer of $u, v: C \to PD$. Then

$$\vdash um = vm,$$

hence

$$my = fx \vdash_{\langle x,y\rangle} ufx = vfx,$$

and therefore, by (8′) of Section 6 (existential specification),

$$\exists_{y\in B}\, my = fx \vdash_x ufx = vfx.$$

But, by supposition,

$$\vdash_x \exists_{y\in B}\, my = fx,$$

hence

$$\vdash_x ufx = vfx,$$

and so, by universal generalization,

$$\vdash \forall_{x\in A}\, ufx = vfx,$$

from which, by Proposition 7.1,

$$uf \cdot{=}\cdot vf.$$

Since m was the equalizer of u and v, the existence of g follows.

PROPOSITION 8.2. *Suppose $m: B \to C$ is a P-regular monomorphism in a dogma, and let*

$$\ulcorner h_m \urcorner \cdot{=}\cdot \{z \in C \mid \exists_{y\in B}\, my = z\}.$$

Then h_m is a characteristic morphism *of m, that is, m is an equalizer of h_m and true 0_C.*

Proof. Note that, for any $f: A \to C$,

$$\begin{aligned}\ulcorner h_m f \urcorner &\cdot{=}\cdot \{x \in A \mid h_m fx\}\\ &\cdot{=}\cdot \{x \in A \mid fx \in \ulcorner h_m \urcorner\}\\ &\cdot{=}\cdot \{x \in A \mid \exists_{y\in B}\, my = fx\}.\end{aligned}$$

Suppose

$$h_m f \cdot{=}\cdot \mathit{true}\, 0_C f \cdot{=}\cdot \mathit{true}\, 0_A\,,$$

then

$$\{x \in A \mid \exists_{y\in B}\, my = fx\} \cdot{=}\cdot \{x \in A \mid \mathit{true}\},$$

hence

$$\vdash \forall_{x\in A}\, \exists_{y\in B}\, my = fx.$$

By Lemma 8.1, there is a unique $g\colon A \to B$ such that $mg \cdot\!=\!\cdot f$, as was to be proved.

THEOREM 8.3. *Let B be an object in a dogma for which the singleton morphism $\iota_B\colon B \to PB$ is a P-regular monomorphism. Suppose $\vdash \forall_{x\in A}\, \exists!_{y\in B}\, p(x, y)$, then there is a unique $g\colon A \to B$ such that $\vdash \forall_{x\in A}\, p(x, gx)$. In fact, $y = gx \cdot\!=\!\cdot_{\langle x,y\rangle}$ $p(x, y)$.*

Proof. By Proposition 8.2, ι_B has a characteristic morphism h_B, where

$$\ulcorner h_B \urcorner \cdot\!=\!\cdot \{z \in PB \mid \exists_{y\in B}\{y\} = z\}.$$

We may assume that

$$p(x, y) \cdot\!=\!\cdot_{\langle x,y\rangle}\, y \in fx,$$

where $f\colon A \to PB$ is given by $fx \cdot\!=\!\cdot_x \{y \in B \mid p(x, y)\}$. Now

$$\begin{aligned} \ulcorner h_B f \urcorner &\cdot\!=\!\cdot \{x \in A \mid \exists_{y\in B}\, \{y\} = fx\} \\ &\cdot\!=\!\cdot \{x \in A \mid \exists!_{y\in B}\, p(x, y)\} \\ &\cdot\!=\!\cdot \{x \in A \mid \mathit{true}\} \\ &\cdot\!=\!\cdot \ulcorner \mathit{true}\, 0_A \urcorner. \end{aligned}$$

Thus

$$h_B f \cdot\!=\!\cdot \mathit{true}\, 0_A\,,$$

hence there exists a unique $g\colon A \to B$ such that

$$\iota_B g \cdot\!=\!\cdot f,$$

therefore,

$$\begin{aligned} p(x, gx) &\cdot\!=\!\cdot_x\, gx \in fx \\ &\cdot\!=\!\cdot_x\, gx \in \{gx\} \\ &\cdot\!=\!\cdot_x\, \mathit{true}. \end{aligned}$$

Thus

$$\vdash_x p(x, gx),$$

and the result follows by universal generalization.

From $\vdash_x p(x, gx)$ we also deduce that $y = gx \vdash_{\langle x,y\rangle} p(x, y)$, and the converse entailment holds since $\vdash_x \exists!_{y\in B}\, p(x, y)$.

When $B = \Omega$ or $B = PC$, Theorem 8.3 may be improved by giving an explicit construction for g.

PROPOSITION 8.4. *Suppose* (a) $B = \Omega$ *or* (b) $B = PC$, and $\vdash \forall_{x\in A}\, \exists!_{y\in B}$ $p(x, y)$. *Then there is a unique $g\colon A \to B$ such that $\vdash \forall_{x\in A}\, p(x, gx)$ given by*

(a) $gx \cdot\!=\!\cdot_x \exists_{t\in\Omega}(t \wedge p(x, t))$,

(b) $gx \cdot\!=\!\cdot_x \{z \in C \mid \exists_{w\in PC}(z \in w \wedge p(x, w))\}$.

In fact, $y = gx \cdot\!=\!\cdot_{\langle x,y\rangle} p(x, y)$.

Proof. Since $\Omega \cong P1$, it is clear that (a) may be obtained as a special case of (b). We shall prove (b).

Define $\sigma_{PC}: P(PC) \to PC$ by

$$\sigma_{PC}s \cdot\!=\!\cdot_s \{z \in C \mid \exists_{w\in PC}\, w \in s \wedge z \in w\},$$

s being an indeterminate of type $P(PC)$. One easily checks that $\sigma_{PC}\iota_{PC} \cdot\!=\!\cdot 1_{PC}$. Hence ι_{PC} is the equalizer of $\iota_{PC}\sigma_{PC}$ and $1_{P(PC)}$ and thus a P-regular monomorphism.

We may now apply Theorem 8.3 and obtain a (unique) g such that $\vdash_x p(x, gx)$. Then clearly

$$z \in gx \vdash_{\langle x,y\rangle} \exists_{w\in PC}(z \in w \wedge p(x, w)).$$

The converse entailment holds also, since $\vdash_x \exists!_{w\in PC}\, p(x, w)$. Therefore

$$z \in gx \cdot\!=\!\cdot_{\langle x,y\rangle} \exists_{w\in PC}(z \in w \wedge p(x, w)).$$

By functional completeness,

$$gx \cdot\!=\!\cdot_x \{z \in C \mid \exists_{w\in PC}(z \in w \wedge p(x, w))\}.$$

9. Top as a Subcategory of Dog

An *elementary topos* is a predogma with a morphism *true*: $1 \to \Omega$ such that

(*) every morphism into Ω is a characteristic morphism of some monomorphism;

(**) every monomorphism has a characteristic morphism;

(***) characteristic morphisms are unique.

We recall that $h: A \to \Omega$ is a *characteristic morphism* of $m: B \to A$ if m is an equalizer of h and *true* 0_A, equivalently, if the following square is a pullback:

$$\begin{tikzcd} A \arrow[r, "h"] & \Omega \\ B \arrow[u, "m"] \arrow[r, "0_B"'] & 1 \arrow[u, "true"'] \end{tikzcd}$$

We did not postulate the existence of equalizers in general; but these may be shown to exist.

Indeed, let $\delta_C: C \times C \to \Omega$ be the characteristic morphism of $\langle 1_C, 1_C\rangle$: $C \to C \times C$.

Suppose $f, g: A \to C$ are such that

$$\delta_C\langle f, g\rangle \cdot\!=\!\cdot\; true\, 0_A .$$

Then there exists a unique $k: A \to C$ such that

$$\langle f, g\rangle \cdot\!=\!\cdot \langle 1_C, 1_C\rangle k \cdot\!=\!\cdot \langle k, k\rangle,$$

hence

$$f \cdot\!=\!\cdot g.$$

Now let $u, v: B \to C$, then $e: A \to B$ equalizes u and v if and only if

$$ue \cdot\!=\!\cdot ve,$$

that is,

$$\delta_C\langle u, v\rangle e \cdot\!=\!\cdot \delta_C\langle ue, ve\rangle \cdot\!=\!\cdot\; true\, 0_A ,$$

that is, e equalizes $\delta_C\langle u, v\rangle$ and $true\, 0_B$. Now an equalizer of the latter two morphisms is assumed to exist by $(*)$, hence u and v have an equalizer.

In view of Remark 7.7, we are led to define $\wedge$, $\Rightarrow$, and $\forall_A$ in a topos as follows:

$$\begin{aligned}
\wedge &\cdot\!=\!\cdot \delta_{\Omega\times\Omega}\langle 1_{\Omega\times\Omega}, \langle true, true\rangle\, 0_{\Omega\times\Omega}\rangle,\\
\Rightarrow &\cdot\!=\!\cdot \delta_{\Omega}\langle \wedge, \pi_{\Omega,\Omega}\rangle,\\
\forall_A &\cdot\!=\!\cdot \delta_{PA}\langle 1_{PA}, (true\, \pi_{1,A})^*\, 0_{PA}\rangle.
\end{aligned}$$

These definitions are easily seen to be equivalent to the usual ones, as stated by Lawvere in his 1972 "Introduction."

It is well-known that a topos satisfies conditions (1) to (8) of a dogma. The axiom of extensionality (9) appears in Fourman [8]. We shall prove a slightly more general result here.

PROPOSITION 9.0. *Extensionality holds in any predogma satisfying conditions* (1) *to* (8) *in which characteristic morphisms are unique.*

Proof. Let

$$\ulcorner\gamma_A\urcorner \cdot\!=\!\cdot \{\langle u, v\rangle \in PA \times PA \mid \forall_{x\in A}(x \in u \Leftrightarrow x \in v)\}.$$

Take any morphisms $f, g\colon B \to PA$, then

$$\gamma_A\langle f, g\rangle \mathrel{\cdot{=}\cdot} \mathit{true}\, 0_B$$

is equivalent to

$$\gamma_A\langle fy, gy\rangle \mathrel{\cdot{=}\cdot_y} \mathit{true},$$

for an indeterminate $y\colon 1 \to B$, in view of functional completeness. By definition of set abstraction, this is equivalent to

$$\vdash_y \forall_{x\in A}\, (x \in fy \Leftrightarrow x \in gy).$$

Now let $f\colon B \to PA$ correspond to $f^+\colon A \times B \to \Omega$, then this may be written

$$\vdash \forall_{\langle x,y\rangle \in A\times B}\, (f^+\langle x, y\rangle \Leftrightarrow g^+\langle x, y\rangle),$$

that is, by Proposition 7.1,

$$f^+\langle x, y\rangle \mathrel{\cdot{=}\cdot_{\langle x,y\rangle}} g^+\langle x, y\rangle.$$

By functional completeness this is the same as $f^+ \mathrel{\cdot{=}\cdot} g^+$, which is equivalent to $f \mathrel{\cdot{=}\cdot} g$. Thus γ_A is a characteristic morphism of $\langle 1_{PA}, 1_{PA}\rangle$, which is a split monomorphism, hence P-regular. If characteristic morphisms are unique, γ_A must coincide with the characteristic morphism predicted by Proposition 8.2, hence

$$\begin{aligned}\ulcorner\gamma_A\urcorner &\mathrel{\cdot{=}\cdot} \{\langle u, v\rangle \in PA \times PA \mid \exists_{w\in PA}\langle 1_{PA}, 1_{PA}\rangle w = \langle u, v\rangle\}\\ &\mathrel{\cdot{=}\cdot} \{\langle u, v\rangle \in PA \times PA \mid u = v\}.\end{aligned}$$

From this the result follows readily.

It is clear from (**) in the definition of a topos that every monomorphism is P-regular. In particular, Proposition 8.2 then gives a construction for the unique characteristic morphism of a monomorphism. It follows that orthodox functors between toposes preserve characteristic morphisms exactly, hence they also preserve equalizers up to isomorphism.

For this reason, toposes and orthodox functors form a full subcategory of Dog.

We should point out that our definition of "elementary topos" differs slightly from that given in [19], inasmuch as we postulate finite products as part of the structure and not just the existence of finite products, and the same goes for powers of Ω.

We shall say that an elementary topos has *canonical subobjects* if to each object A, there is associated a representative set Sub A of monomorphisms $B \to A$ with the following properties:

(i) Every monomorphism $B \to A$ is isomorphic to exactly one element of Sub A.

(ii) $1_A: A \to A$ is in Sub A.

(iii) If $f: B \to A$ is in Sub A and $g: D \to C$ is in Sub C, then $f \times g: B \times D \to A \times C$ is in Sub$(A \times C)$.

(iv) If $f: B \to A$ is in Sub A, then $Pf: PB \to PA$ is in Sub(PA).

(v) If $f: B \to A$ is in Sub A and $g: C \to B$ is in Sub B, then $gf: C \to A$ is in Sub A.

We point out that already in a Cartesian category one may define

$$f \times g \cdot\!=\!\cdot \langle f\pi_{B,D}, g\pi'_{B,D}\rangle,$$

and in a dogma one may define Pf as the unique morphism such that, for an indeterminate v of type PB,

$$(Pf)v \cdot\!=\!\cdot_v \{x \in A \mid \exists_{y \in B} (fy = x \wedge y \in v)\}.$$

Thus P is here regarded as a covariant functor $\mathfrak{A} \to \mathfrak{A}$.

In a topos with canonical subobjects there is a bijection

$$\text{Sub } A \underset{\text{ker}}{\overset{\text{char}}{\rightleftarrows}} \text{Hom}(A, \Omega),$$

where

$$\text{char } m = \text{characteristic morphism of } m$$

and

$$\begin{aligned}\ker m &= \text{kernel of } m \\ &= \text{the element of Sub } A \text{ which is an equalizer of } h \text{ and } true\ 0_A .\end{aligned}$$

We write Ker m for the domain of ker m.

We record the following for later use.

LEMMA 9.1. *In a topos with canonical subobjects, the following equations hold:*

(i) $\ker(true\ 0_A) \cdot\!=\!\cdot 1_A$,

(ii) $\ker(\wedge(f \times g)) \cdot\!=\!\cdot \ker f \times \ker g$,

(iii) $\ker(\mathbf{P}\beta^{\complement}) \cdot\!=\!\cdot P(\ker(\beta^{\complement}))$,

where $f: A \to \Omega$, $g: B \to \Omega$, $\beta: 1 \to PB$ *and*

$$\mathbf{P}\beta \cdot\!=\!\cdot \{v \in PB \mid v \subseteq \beta\}.$$

Here (iii) depends on extensionality.

Proof. (i) Since

$$\begin{tikzcd} A \arrow[r, "true\, 0_A"] & \Omega \\ A \arrow[u, "1_A"] \arrow[r, "0_A"'] & 1 \arrow[u, "true"'] \end{tikzcd}$$

is a pullback, we have

$$1_A \cong \ker(true\, 0_A).$$

Since 1_A is in Sub A, the isomorphism may be replaced by equality.

(ii) Consider the following diagram:

$$\begin{tikzcd} A \times B \arrow[r, "f\times g"] & \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \\ \mathrm{Ker}\, f \times \mathrm{Ker}\, g \arrow[u, "\ker f \times \ker g"] \arrow[r, "0_{\mathrm{Ker} f} \times 0_{\mathrm{Ker} g}"'] & 1 \times 1 \arrow[u, "true \times true"'] \arrow[r, "0_{1\times 1}"'] & 1 \arrow[u, "true"'] \end{tikzcd}$$

The square on the left is a Cartesian product of two pullbacks, hence is itself a pullback. The square on the right is isomorphic to the pullback

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \\ 1 \arrow[u, "\langle true,\, true\rangle"] \arrow[r] & 1 \arrow[u, "true"'] \end{tikzcd}$$

which serves as the definition of $\wedge$ in a topos. As is well-known, the juxtaposition of two pullbacks yields a pullback, hence

$$\ker(\wedge(f \times g)) \cong \ker f \times \ker g.$$

But $\ker f \times \ker g$ is in Sub A, hence the isomorphism may be replaced by equality.

To prove (iii), it suffices to show that

$$(\mathbf{P}\beta)^{(} \cdot\!=\!\cdot\ \mathrm{char}(P(\ker(\beta^{(}))).$$

Put $\ker(\beta^{(}) \cdot\!=\!\cdot\ m$, then $\beta^{(} \cdot\!=\!\cdot\ \mathrm{char}\, m$, so we want to show that

(1) $\mathbf{P}(\ulcorner \mathrm{char}\, m \urcorner) \cdot\!=\!\cdot\ \ulcorner \mathrm{char}(Pm) \urcorner$.

Now, the left side of (1) is

$$\{v \in PB \mid v \subseteq \ulcorner \mathrm{char}\, m \urcorner\},$$

and, in view of Proposition 8.2, this is

$$\{v \in PB \mid v \subseteq \{y \in B \mid \exists_{z \in C}\, mz = y\}\}.$$

On the other hand, Pm is defined to be the unique morphism such that, for an indeterminate w of type PC,

$$(Pm)w \cdot\!=\!\cdot_w \{y \in B \mid \exists_{z \in C}\, (mz = y \wedge z \in w)\},$$

hence, the right-hand side of (1) is, in view of Proposition 8.2,

$$\begin{aligned}&\{v \in PB \mid \exists_{w \in PC}\, (Pm)w = v\}\\ &\quad \cdot\!=\!\cdot\ \{v \in PB \mid \exists_{w \in PC}\, \{y \in B \mid \exists_{z \in C}\, mz = y \wedge z \in w\} = v\}.\end{aligned}$$

Thus we want to prove the equivalence in $\mathcal{A}[v]$ of the following two propositions (in view of extensionality):

(2) $\forall_{y \in B}\, (y \in v \Rightarrow \exists_{z \in C}\, mz = y)$,

(3) $\exists_{w \in PC}\, \forall_{y \in B}\, (y \in v \Leftrightarrow \exists_{z \in C}\, (mz = y \wedge z \in w))$.

That (3) implies (2) is an exercise in the predicate calculus. To show that (2) implies (3), let us assume (2) and put

$$m^{-1}v \cdot\!=\!\cdot_v \{z \in C \mid mz \in v\},$$

then we deduce

$$y \in v \Rightarrow \exists_{z \in C}\, (mz = y \wedge z \in m^{-1}v).$$

But, since

$$z \in m^{-1}v \Leftrightarrow mz \in v,$$

we also have the converse implication, hence

$$\forall_{y \in B}\, (y \in v \Leftrightarrow \exists_{z \in C}\, (mz = y \wedge z \in m^{-1}v)),$$

from which (3) follows by existential generalization.

We are interested in forming a category Top whose objects are elementary toposes with canonical subobjects. What are the morphisms of this category? They are of course the usual "logical morphisms" (suitably canonified) and are defined by the equivalent conditions of the following:

LEMMA 9.2. *Let $G: \mathcal{A} \to \mathcal{B}$ be a functor between elementary toposes with canonical subobjects. Then the following two statements are equivalent*:

(a) *G is orthodox and preserves canonical subobjects*;

(b) *G is preorthodox and preserves canonical kernels.*

Proof. Suppose (a), then surely G is preorthodox. In view of Proposition 8.2, G preserves characteristic morphisms, hence it preserves canonical kernels, since ker is inverse to char.

Suppose (b), then G preserves characteristic morphisms, since char is inverse to ker. Hence G preserves *true*, $\wedge$, $\Leftrightarrow$, and $\forall$, in view of their usual definitions as characteristic morphisms of $1_1: 1 \to 1$, $\langle true, true\rangle: 1 \to \Omega \times \Omega$, $\langle 1_\Omega, 1_\Omega\rangle: \Omega \to \Omega \times \Omega$, and $\ulcorner true\, 0_A \urcorner: 1 \to PA$, respectively. Moreover, G preserves $\Rightarrow$, which can be defined in terms of $\wedge$ and $\Leftrightarrow$, and it preserves *false*, $\vee$, and $\exists$, in view of Remark 7.5.

We note that Top is a subcategory of Dog, but not a full subcategory.

10. The Topos Associated with a Dogma

If α and β are sets of types PA and PB respectively, we write

$$\alpha \times \beta \cdot\!=\!\cdot \{\langle x, y\rangle \in A \times B \mid x \in \alpha \wedge y \in \beta\}.$$

A *relation* $\rho: \beta \to \alpha$ is a triple $(\beta, |\rho|, \alpha)$, where $|\rho| \leqslant \alpha \times \beta$. Thus $|\rho|$ is a set of type $P(A \times B)$. In particular, we have the *identity* relation $1_\alpha: \alpha \to \alpha$, the *converse* $\rho^{-1}: \alpha \to \beta$ of $\rho: \beta \to \alpha$, and the *relative product* $\rho * \sigma: \gamma \to \alpha$ of $\rho: \beta \to \alpha$ and $\sigma: \gamma \to \beta$, where

$$\begin{aligned} |1_\alpha| &\cdot\!=\!\cdot \{\langle x, x\rangle \in A \times A \mid x \in \alpha\}, \\ |\rho^{-1}| &\cdot\!=\!\cdot \{\langle y, x\rangle \in B \times A \mid \langle x, y\rangle \in |\rho|\}, \\ |\rho * \sigma| &\cdot\!=\!\cdot \{\langle x, z\rangle \in A \times C \mid \exists_{y \in B} \langle x, y\rangle \in |\rho| \wedge \langle y, z\rangle \in |\sigma|\}. \end{aligned}$$

Sets and relations form a category. We are interested in the subcategory of sets and functions, where $\rho: \beta \to \alpha$ is called a *function* if

$$\vdash \forall_{y \in B} (y \in \beta \Rightarrow \exists!_{x \in A} \langle x, y\rangle \in |\rho|),$$

that is, if

$$|1_\beta| \leqslant |\rho^{-1} * \rho|, \qquad |\rho * \rho^{-1}| \leqslant |1_\alpha|.$$

Lemma 10.1 *A function $\rho: \beta \to \alpha$ is a monomorphism if and only if $\rho^{-1} * \rho \cdot\!=\!\cdot 1_\beta$.*

Proof. Assuming this condition and $\rho * \sigma \cdot\!=\!\cdot \rho * \tau$, we immediately deduce $\sigma \cdot\!=\!\cdot \tau$. Conversely, suppose ρ is mono. Consider the set

$$\begin{aligned} \gamma &\cdot\!=\!\cdot |\rho^{-1} * \rho| \\ &\cdot\!=\!\cdot \{\langle y, y'\rangle \in B \times B \mid \exists_{x \in A} \langle x, y\rangle \in |\rho| \wedge \langle x, y'\rangle \in |\rho|\}. \end{aligned}$$

Define $\sigma, \tau: \gamma \to \beta$ by

$$|\sigma| \cdot\!=\!\cdot \{\langle y, \langle y, y'\rangle\rangle \in B \times (B \times B) \mid \langle y, y'\rangle \in \gamma\},$$
$$|\tau| \cdot\!=\!\cdot \{\langle y', \langle y, y'\rangle\rangle \in B \times (B \times B) \mid \langle y, y'\rangle \in \gamma\}.$$

Then

$$|\rho * \sigma| \cdot\!=\!\cdot \{\langle x, \langle y, y'\rangle\rangle \in A \times (B \times B) \mid \langle x, y\rangle \in |\rho| \wedge \langle y, y'\rangle \in \gamma\},$$
$$|\rho * \tau| \cdot\!=\!\cdot \{\langle x, \langle y, y'\rangle\rangle \in A \times (B \times B) \mid \langle x, y'\rangle \in |\rho| \wedge \langle y, y'\rangle \in \gamma\}.$$

Since ρ is a function, it follows from the definition of γ that $\rho * \sigma \cdot\!=\!\cdot \rho * \tau$, hence that $\sigma \cdot\!=\!\cdot \tau$. Therefore $\gamma \leqslant |1_\beta|$, hence $\gamma \cdot\!=\!\cdot |1_\beta|$.

The following result is due to Volger, but we give our own proof.

PROPOSITION 10.2 *The sets and functions of a dogma form a topos.*

Proof. To obtain the Cartesian structure we define $\mathbf{0}_\alpha: \alpha \to \mathbf{1}$, $\boldsymbol{\pi}_{\alpha,\beta}: \alpha \times \beta \to \alpha$, $\boldsymbol{\pi}_{\alpha,\beta}: \alpha \times \beta \to \beta$ as follows:

$$\mathbf{1} \cdot\!=\!\cdot \{x \in 1 \mid \mathit{true}\} = \{0_1\},$$
$$|\mathbf{0}_\alpha| \cdot\!=\!\cdot \{0_1\} \times \alpha = \langle 0_1, x\rangle \in 1 \times A \mid x \in \alpha\},$$
$$|\boldsymbol{\pi}_{\alpha,\beta}| \cdot\!=\!\cdot \{\langle x, \langle x, y\rangle\rangle \in A \times (A \times B) \mid x \in \alpha \wedge y \in \beta\},$$
$$|\boldsymbol{\pi}'_{\alpha,\beta}| \cdot\!=\!\cdot \{\langle y, \langle x, y\rangle\rangle \in B \times (A \times B) \mid x \in \alpha \wedge y \in \beta\}.$$

To these should be added the rule

$$\frac{\varphi: \gamma \to \alpha \;\; \psi: \gamma \to \beta}{\boldsymbol{\langle} \varphi, \psi \boldsymbol{\rangle}: \gamma \to \alpha \times \beta},$$

where

$$|\boldsymbol{\langle} \varphi, \psi \boldsymbol{\rangle}| \cdot\!=\!\cdot \{\langle\langle x, y\rangle, z\rangle \in (A \times B) \times C \mid \langle x, z\rangle \in |\varphi| \wedge \langle y, z\rangle \in |\psi|\}.$$

The equations of a Cartesian category are easily verified.

To obtain the structure of a predogma, we define $\boldsymbol{\Omega}$, $\mathbf{P}\beta$, $\boldsymbol{\epsilon}_\beta: \mathbf{P}\beta \times \beta \to \boldsymbol{\Omega}$, and

$$\frac{\varphi: \alpha \times \beta \to \boldsymbol{\Omega}}{\varphi^*_-: \alpha \to \mathbf{P}\beta}$$

as follows:

$$\boldsymbol{\Omega} \cdot\!=\!\cdot \{t \in \Omega \mid \mathit{true}\},$$
$$\mathbf{P}\beta \cdot\!=\!\cdot \{z \in PB \mid z \subseteq \beta\},$$
$$|\boldsymbol{\epsilon}_\beta| \cdot\!=\!\cdot \{\langle t, \langle z, y\rangle\rangle \in \Omega \times (PB \times B) \mid z \subseteq \beta \wedge y \in \beta \wedge t = (y \in z)\}$$
$$|\varphi^*_-| \cdot\!=\!\cdot \{\langle z, x\rangle \in PB \times A \mid z = \{y \in B \mid \exists_{t \in \Omega}(t \wedge \langle t, \langle x, y\rangle\rangle \in |\varphi|)\}\}.$$

The two equations of a predogma are easily verified using Proposition 8.4.

Finally, we define **true**: $\mathbf{1} \to \mathbf{\Omega}$ by

$$|\,\mathbf{true}\,| \cdot\!=\!\cdot \{\langle \mathbf{true}, 0_1\rangle\},$$

and it remains to check (∗) to (∗∗∗) in the definition of an elementary topos.

(∗) We claim that $\chi\colon \alpha \to \mathbf{\Omega}$ is a characteristic function of $\ker \chi\colon \mathrm{Ker}\,\chi \to \alpha$, where

$$\mathrm{Ker}\,\chi \cdot\!=\!\cdot \{x \in A \mid \exists_{t\in\Omega} (t \wedge \langle t, x\rangle \in |\,\chi\,|)\},$$
$$|\ker \chi\,| \cdot\!=\!\cdot \{\langle x, x\rangle \in A \times A \mid \exists_{t\in\Omega} (t \wedge \langle t, x\rangle \in |\,\chi\,|)\}.$$

Indeed, suppose $\rho\colon \beta \to \alpha$ is such that $\chi * \rho \cdot\!=\!\cdot \mathbf{true} * \mathbf{0}_\beta$, then

$$|\,\chi * \rho\,| \cdot\!=\!\cdot \{\langle \mathit{true}, y\rangle \in \Omega \times B \mid y \in \beta\}.$$

We may define $\xi\colon \beta \to \mathrm{Ker}\,\chi$ by

$$\xi \cdot\!=\!\cdot (\beta, |\,\rho\,|, \mathrm{Ker}\,\chi).$$

This is easily seen to be the unique function for which

$$\ker \chi * \xi \cdot\!=\!\cdot \rho.$$

(∗∗) Suppose $\rho\colon \beta \to \alpha$ is a given monomorphism. We claim that it has a characteristic function $\mathrm{char}\,\rho\colon \alpha \to \mathbf{\Omega}$, where

$$|\,\mathrm{char}\,\rho\,| \cdot\!=\!\cdot \{\langle t, x\rangle \in \Omega \times A \mid t = \exists_{y\in B} \langle x, y\rangle \in |\,\rho\,|\}.$$

Indeed, define $\xi\colon \beta \to \mathrm{Ker\,char}\,\rho$ as above by $|\,\xi\,| \cdot\!=\!\cdot |\,\rho\,|$, then

$$\begin{aligned} |\,\xi^{-1} * \xi\,| &\cdot\!=\!\cdot |\,\rho^{-1} * \rho\,| \\ &\cdot\!=\!\cdot |\,1_\beta\,|, \end{aligned}$$

by Lemma 10.1. On the other hand,

$$\begin{aligned} |\,\xi * \xi^{-1}\,| &\cdot\!=\!\cdot |\,\rho * \rho^{-1}\,| \\ &\cdot\!=\!\cdot \{\langle x, x\rangle \in A \times A \mid \exists_{y\in B} \langle x, y\rangle \in |\,\rho\,|\} \\ &\cdot\!=\!\cdot \{\langle x, x\rangle \in A \times A \mid \exists_{t\in\Omega} (t \wedge \langle t, x\rangle \in |\,\mathrm{char}\,\rho\,|)\} \\ &\cdot\!=\!\cdot \{\langle x, x\rangle \in A \times A \mid x \in \mathrm{Ker\,char}\,\rho\} \\ &\cdot\!=\!\cdot |\,1_{\mathrm{Ker\,char}\,\rho}\,|. \end{aligned}$$

Therefore, ξ is an isomorphism, from which it follows that char ρ is a characteristic function not only of ker char ρ but also of ρ,

(***) Suppose $\chi: \alpha \to \Omega$ is a characteristic function of the monomorphism $\rho: \beta \to \alpha$. We claim that $\chi \cdot\!=\!\cdot$ char ρ.

Indeed, since equalizers in a category are unique up to isomorphism, we see that the function ξ defined in (*) is an isomorphism. Therefore,

$$\begin{aligned} | \rho * \rho^{-1} | &\cdot\!=\!\cdot | \xi * \xi^{-1} | \\ &\cdot\!=\!\cdot | 1_{\mathrm{Ker}\chi} | \\ &\cdot\!=\!\cdot \{\langle x, x\rangle \in A \times A \mid \exists_{t\in\Omega} (t \wedge \langle t, x\rangle \in | \chi |)\}. \end{aligned}$$

In other words,

$$\exists_{y\in B} \langle x, y\rangle \in | \rho | \cdot\!=\!\cdot_x \exists_{t\in\Omega} (t \wedge \langle t, x\rangle \in | \chi |),$$

hence

$$\begin{aligned} | \operatorname{char} \rho | &\cdot\!=\!\cdot \{\langle t, x\rangle \in \Omega \times A \mid t = \exists_{y\in B} \langle x, y\rangle \in | \rho |\} \\ &\cdot\!=\!\cdot \{\langle t, x\rangle \in \Omega \times A \mid t = \exists_{t'\in\Omega} (t' \wedge \langle t', x\rangle \in | \chi |)\} \\ &\cdot\!=\!\cdot \{\langle t, x\rangle \in \Omega \times A \mid \langle t, x\rangle \in | \chi |\}, \end{aligned}$$

by Proposition 8.4, and so char $\rho \cdot\!=\!\cdot \chi$, as was to be proved.

PROPOSITION 10.3. *The topos associated with a dogma has canonical subobjects.*

Proof. If α is any set of type PA, Sub α consists of all sets α' of the same type such that $\alpha' \leqslant \alpha$, together with the inclusion $\xi: \alpha' \to \alpha$, where $| \xi | = \{\langle x, x\rangle \in A \times A \mid x \in \alpha'\}$. In the remainder of this proof, we mostly ignore ξ and regard α' itself as the subobject of α, as is usually done. We shall check conditions (i) to (v) of the definition of canonical subobjects in Section 9.

(i) Let $\mu: \beta \to \alpha$ be a monomorphism in the topos associated with the dogma $\mathfrak{A}$, where $\alpha: 1 \to PA$ and $\beta: 1 \to PB$. We claim that there is a subobject $\xi: \alpha' \to \alpha$ and an isomorphism ρ such that $\xi * \rho \cdot\!=\!\cdot \mu$. Indeed, let

$$\alpha' \cdot\!=\!\cdot \{x \in A \mid \exists_{y\in B} \langle x, y\rangle \in | \mu |\}$$

and

$$| \rho | \cdot\!=\!\cdot \{\langle x, y\rangle \in A \times B \mid \langle x, y\rangle \in | \mu |\}.$$

It is easily seen that ρ is an isomorphism (see the proof that ξ is an isomorphism in condition (**) of Proposition 10.2).

(ii) Clearly $\alpha \leqslant \alpha$.

(iii) If $\alpha' \leqslant \alpha$ and $\beta' \leqslant \beta$, then $\alpha' \times \beta' \leqslant \alpha \times \beta$, as

$$\{\langle x, y\rangle \in A \times B \mid x \in \alpha' \wedge y \in \beta'\} \leqslant \{\langle x, y\rangle \in A \times B \mid x \in \alpha \wedge y \in \beta\}.$$

(iv) If $\alpha' \leqslant \alpha$, then $\mathbf{P}\alpha' \leqslant \mathbf{P}\alpha$. Indeed,

$$\begin{aligned} \mathbf{P}\alpha' &\cdot\!=\!\cdot \{x \in PA \mid x \subseteq \alpha'\} \\ &\leqslant \{x \in PA \mid x \subseteq \alpha\} \\ &\cdot\!=\!\cdot \mathbf{P}\alpha. \end{aligned}$$

(v) If $\alpha' \leqslant \alpha$ and $\alpha'' \leqslant \alpha'$, then clearly $\alpha'' \leqslant \alpha$.

As far as I can see, none of the results in Section 10 depend on the axiom of extensionality.

11. Top as a Reflective Subcategory of Dog

Let $T(\mathfrak{A})$ be the topos associated with the dogma $\mathfrak{A}$. We shall exhibit an orthodox functor $H: \mathfrak{A} \to T(\mathfrak{A})$ which will be seen to have the expected universal property.

We define H on objects by

$$H(A) \cdot\!=\!\cdot \{x \in A \mid true\}.$$

It is immediately seen that then

$$H(1) \cdot\!=\!\cdot \mathbf{1}, \qquad H(A \times B) \cdot\!=\!\cdot H(A) \times H(B),$$
$$H(\Omega) \cdot\!=\!\cdot \mathbf{\Omega}, \qquad H(PA) \cdot\!=\!\cdot \mathbf{P}H(A).$$

We define H on morphisms $f: A \to B$ by

$$\begin{aligned} | H(f)| &\cdot\!=\!\cdot graph\, f \\ &\cdot\!=\!\cdot \{\langle y, x\rangle \in B \times A \mid y = fx\}. \end{aligned}$$

One easily checks that

$$H(1_A) = 1_{H(A)}, \qquad H(gf) = H(g) * H(f),$$
$$H(0_A) = \mathbf{0}_{H(A)},$$

etc. Thus H is a preorthodox functor. To show that it is orthodox, we utilize Remark 7.7 and verify that it preserves *true* and $\doteq$. (It is here that extensionality comes in.)

Indeed, it is easily seen that $| H(true)| \cdot\!=\!\cdot | \mathbf{true} |$. To see that H preserves $=$, we must check that $H(\delta_A)$ is the usual equality in the topos $T(\mathfrak{A})$, that is, the characteristic morphism of the diagonal $\langle 1_{H(A)}, 1_{H(A)} \rangle: H(A) \to H(A) \times H(A)$. This is a consequence of the following two statements, whose verification is left to the reader:

(i) $H(\delta_A) * \langle 1_{H(A)}, 1_{H(A)} \rangle \cdot\!=\!\cdot\ \mathbf{true} * \mathbf{0}_{H(A)}$;

(ii) if $H(\delta_A) * \langle \rho, \sigma \rangle \cdot\!=\!\cdot\ \mathbf{true} * \mathbf{0}_B$, then

$$\langle \rho, \sigma \rangle \cdot\!=\!\cdot \langle 1_{H(A)}, 1_{H(A)} \rangle * \rho.$$

THEOREM 11.1. *For any dogma $\mathcal{A}$ (with extensionality) there exists an orthodox functor $H: \mathcal{A} \to T(\mathcal{A})$ such that, for any topos $\mathcal{E}$ and any orthodox functor $G: \mathcal{A} \to \mathcal{E}$, there exists a unique orthodox functor $G': T(\mathcal{A}) \to \mathcal{E}$ which preserves canonical subobjects and such that $G'H = G$.*

In other words: T is left adjoint to the forgetful functor from Top (with subobject preserving orthodox functors as morphisms) to Dog, and H is the adjunction.

$$\begin{tikzcd} T(\mathcal{A}) \arrow[rr, dashed, "G'"] & & \mathcal{E} \\ & \mathcal{A} \arrow[ul, "H"] \arrow[ur, "G"'] & \end{tikzcd}$$

Proof. How to define G' on objects? Let α be any set of type PA. The inclusion of α into $H(A)$ is given by the monomorphism $\varphi_\alpha: \alpha \to H(A)$, where

$$|\varphi_\alpha| \cdot\!=\!\cdot \{\langle x, x \rangle \in A \times A \mid x \in \alpha\}.$$

φ_α has characteristic function $\mathrm{char}\, \varphi_\alpha: H(A) \to \mathbf{\Omega}$, where

$$\begin{aligned} |\mathrm{char}\, \varphi_\alpha| &\cdot\!=\!\cdot \{\langle t, x \rangle \in \Omega \times A \mid t = \exists_{x' \in A} \langle x, x' \rangle \in |\varphi_\alpha|\} \\ &\cdot\!=\!\cdot \{\langle t, x \rangle \in \Omega \times A \mid t = (x \in \alpha)\}. \end{aligned}$$

Let $\alpha: 1 \to PA$ correspond to $\alpha^{(}: A \to \Omega$, then this may be written

$$|\mathrm{char}\, \varphi_\alpha| \cdot\!=\!\cdot\ \mathit{graph}\, \alpha^{(},$$

hence

$$\mathrm{char}\, \varphi_\alpha \cdot\!=\!\cdot H(\alpha^{(}).$$

Moreover

$$\begin{aligned} \mathrm{Ker}\, H(\alpha^{(}) &\cdot\!=\!\cdot \{x \in A \mid \exists_{t \in \Omega} (t \wedge \langle t, x \rangle \in |\mathrm{char}\, \varphi_\alpha|)\} \\ &\cdot\!=\!\cdot \{x \in A \mid \exists_{t \in \Omega} (t \wedge t = (x \in \alpha))\} \\ &\cdot\!=\!\cdot \{x \in A \mid x \in \alpha\} \\ &\cdot\!=\!\cdot\ \alpha, \end{aligned}$$

and similarly

$$\ker H(\alpha^{\langle}) \cdot\!=\!\cdot \varphi_\alpha .$$

Since G' is supposed to preserve kernels, we must have

$$\begin{aligned} G'(\varphi_\alpha) &\cdot\!=\!\cdot \ker G'H(\alpha^{\langle}) \\ &\cdot\!=\!\cdot \ker G(\alpha^{\langle}) \\ &\cdot\!=\!\cdot \mu_\alpha , \qquad \text{say.} \end{aligned}$$

We are thus forced to define $G'(\alpha)$ as *the domain of the kernel of* $G(\alpha^{\langle})$.

How to define G' on morphisms? Consider a function $\rho: \beta \to \alpha$, we shall define $G'(\rho)$ as the unique morphism h: $G'(\beta) \to G'(\alpha)$ such that

$$\text{(i)} \quad \vdash \forall_{y \in G'(\beta)} \langle \mu_\alpha h y, \mu_\beta y \rangle \in G(|\,\rho\,|).$$

To justify this definition, we shall prove

(a) that there is a unique h satisfying (i),

(b) that G' so defined is a preorthodox functor which preserves canonical kernels (see Lemma 9.2) and that $G'H = G$,

(c) if G' is an orthodox functor such that $G'H = G$, then $h \cdot\!=\!\cdot G'(\rho)$ must satisfy (i).

(a) By definition of μ_β, we have

$$G(\beta)^{\langle} \mu_\beta \cdot\!=\!\cdot \mathit{true}\, 0_{G(B)} .$$

It follows that

$$\vdash \forall_{y \in G'(\beta)} G(\beta)^{\langle} \mu_\beta y = \mathit{true},$$

that is,

$$\text{(ii)} \quad \vdash \forall_{y \in G'(\beta)} \mu_\beta y \in G(\beta).$$

But, since ρ is a function,

$$\vdash \forall_{y \in B} (y \in \beta \Rightarrow \exists!_{x \in A} \langle x, y \rangle \in |\,\rho\,|).$$

Applying the orthodox functor G to this, we obtain

$$\vdash \forall_{y \in G(B)} (y \in G(\beta) \Rightarrow \exists!_{x \in G(A)} \langle x, y \rangle \in G(|\,\rho\,|)).$$

Hence, by universal specification, for y of type $G'(\beta)$,

$$\vdash_y \mu_\beta y \in G(\beta) \Rightarrow \exists!_{x \in G(A)} \langle x, \mu_\beta y \rangle \in G(|\,\rho\,|).$$

Therefore, in view of (ii), we have

$$\vdash \forall_{y \in G'(\beta)} \exists!_{x \in G(A)} \langle x, \mu_\beta y \rangle \in G(|\, \rho \,|).$$

By Theorem 8.3, there exists a unique $g\colon G'(\beta) \to G(A)$ such that

$$\text{(iii)} \quad \vdash \forall_{y \in G'(\beta)} \langle gy, \mu_\beta y \rangle \in G(|\, \rho \,|).$$

Now

$$\vdash_y gy \in G(A),$$

hence

$$\begin{aligned} G(\alpha^{\complement})\, gy &\cdot\!=\!\cdot_y\, G(\alpha)^{\complement}\, gy \\ &\cdot\!=\!\cdot_y\, \mathit{true} \\ &\cdot\!=\!\cdot_y\, \mathit{true}\, 0_{G'(\beta)}\, y, \end{aligned}$$

and therefore

$$G(\alpha^{\complement})g \cdot\!=\!\cdot\, \mathit{true}\, 0_{G'(\beta)}\,.$$

Since $\mu_\alpha\colon G'(\alpha) \to G(A)$ is the kernel of $G(\alpha^{\complement})$, there exists a unique $h\colon G'(\beta) \to G'(\alpha)$ such that $\mu_\alpha h \cdot\!=\!\cdot g$. It then follows from (iii) that h satisfies (i). Moreover, since μ_α is a monomorphism, h is unique with this property.

(b) For any object A of $\mathcal{Q}$, we have

$$\begin{aligned} G'H(A) &\cdot\!=\!\cdot\, \operatorname{Ker} G(H(A)^{\complement}) \\ &\cdot\!=\!\cdot\, \operatorname{Ker} G(\mathit{true}\, 0_A) \\ &\cdot\!=\!\cdot\, \operatorname{Ker}(\mathit{true}\, 0_{G(A)}) \\ &\cdot\!=\!\cdot\, G(A), \end{aligned}$$

by Lemma 9.1(i).

It follows that $G'(\mathbf{1}) \cdot\!=\!\cdot 1$ and $G'(\mathbf{\Omega}) \cdot\!=\!\cdot \Omega$. Moreover,

$$\begin{aligned} G'(\alpha \boldsymbol{\times} \beta) &\cdot\!=\!\cdot\, \operatorname{Ker} G((\alpha \boldsymbol{\times} \beta)^{\complement}) \\ &\cdot\!=\!\cdot\, \operatorname{Ker} G(\wedge(\alpha^{\complement} \times \beta^{\complement})) \\ &\cdot\!=\!\cdot\, \operatorname{Ker}(\wedge(G(\alpha^{\complement}) \times G(\beta^{\complement}))) \\ &\cdot\!=\!\cdot\, \operatorname{Ker} G(\alpha^{\complement}) \times \operatorname{Ker} G(\beta^{\complement}), \end{aligned}$$

by Lemma 91(ii). Furthermore,

$$\begin{aligned} G'(\mathbf{P}\alpha) &\cdot\!=\!\cdot\, \operatorname{Ker} G((\mathbf{P}\alpha)^{\complement}) \\ &\cdot\!=\!\cdot\, G(\operatorname{Ker}((\mathbf{P}\alpha)^{\complement})) \\ &\cdot\!=\!\cdot\, GP \operatorname{Ker}(\alpha^{\complement}) \\ &\cdot\!=\!\cdot\, PG \operatorname{Ker}(\alpha^{\complement}) \\ &\cdot\!=\!\cdot\, PG'(\alpha), \end{aligned}$$

by Lemma 9.1(iii).

Next, we shall show that G' preserves canonical kernels.

Suppose $\chi: \alpha \to \Omega$ has kernel $\rho: \beta \to \alpha$, then an easy calculation yields $|\rho| \cdot\!=\!\cdot |\varphi_\beta|$.

Since $|\operatorname{char} \rho|$ depends only on $|\rho|$, we have $|H(\beta^{\text{(}})| \cdot\!=\!\cdot |\operatorname{char} \varphi_\beta| \cdot\!=\!\cdot |\operatorname{char} \rho| \cdot\!=\!\cdot |\chi|$. Therefore also $\varphi_\alpha * \rho \cdot\!=\!\cdot \varphi_\beta$ and $H(\beta^{\text{(}}) * \varphi_\alpha \cdot\!=\!\cdot \chi$, the absolute values having been omitted since sources and targets agree. Applying G' to these two equations, one obtains $\mu_\alpha G'(\rho) \cdot\!=\!\cdot \mu_\beta$ and $G(\beta^{\text{(}})\mu_\alpha \cdot\!=\!\cdot G'(\chi)$. It is now routine to verify that $\mu_\alpha \ker G'(\chi)$ has characteristic morphism $G(\beta^{\text{(}})$. Since μ_α and $\ker G'(\chi)$ are both canonical subobjects, so is their composition, hence $\mu_\alpha \ker G'(\chi) \cdot\!=\!\cdot \ker G(\beta^{\text{(}}) \cdot\!=\!\cdot \mu_\beta \cdot\!=\!\cdot \mu_\alpha G'(\rho)$. Since $\mu_\alpha \cdot\!=\!\cdot \ker G(\alpha^{\text{(}})$ is a monomorphism, we have $\ker G'(\chi) \cdot\!=\!\cdot G'(\rho) \cdot\!=\!\cdot G'(\ker \chi)$, as was to be shown.

So far we have only discussed the effect of G' on objects of $T(\mathcal{C})$. In view of Lemma 9.2, the following equations remain to be checked:

$$
\begin{aligned}
G'(1_\alpha) &\cdot\!=\!\cdot 1_{G'(\alpha)}\,, \\
G'(\varphi * \psi) &\cdot\!=\!\cdot G'(\varphi)\, G'(\psi) \text{ for functions } \varphi \text{ and } \psi, \\
G'H(f) &\cdot\!=\!\cdot G(f), \\
G'(0_\alpha) &\cdot\!=\!\cdot 0_{G'(\alpha)}\,, \\
G'(\pi_{\alpha,\beta}) &\cdot\!=\!\cdot \pi_{G'(\alpha),G'(\beta)}\,, \text{ and similarly for } \pi'_{\alpha,\beta}\,, \\
G'\langle\varphi, \psi\rangle &\cdot\!=\!\cdot \langle G'(\varphi), G'(\psi)\rangle, \\
G'(\varphi^*_-) &\cdot\!=\!\cdot G'(\varphi)^*, \\
G'(\epsilon_\beta) &\cdot\!=\!\cdot \epsilon_{G'(\beta)}\,.
\end{aligned}
$$

Now all these equations have the form

$$G'(\rho) \cdot\!=\!\cdot h.$$

Therefore, in view of (a), we only have to check the appropriate instance of (i).

For example, to prove the third equation with $f: A \to B$, we want to check that

$$\vdash \forall_{x \in G'H(A)} \langle \mu_{H(B)} G'H(f)x, \mu_{H(A)}x\rangle \in G(|H(f)|),$$

that is

$$\vdash \forall_{x \in G(A)} \langle G(f)x, x\rangle \in G(|H(f)|),$$

since

$$
\begin{aligned}
\mu_{H(A)} &\cdot\!=\!\cdot G'(\varphi_{H(A)}) \cdot\!=\!\cdot G'(1_{H(A)}) \\
&\cdot\!=\!\cdot 1_{G'H(A)} \cdot\!=\!\cdot 1_{G(A)}\,,
\end{aligned}
$$

in view of the first equation, which we assume to have been proved. Now

$$\begin{aligned} G(|\,H(f)|) &\cdot\!=\!\cdot\; G(\{\langle y, x\rangle \in B \times A \mid y = fx\}) \\ &\cdot\!=\!\cdot\; \{\langle y, x\rangle \in G(B) \times G(A) \mid y = G(f)x\}, \end{aligned}$$

in view of Corollary 5.4. Hence

$$\vdash_x \langle G(f)x, x\rangle \in G(|\,H(f)|),$$

from which the result follows by universal generalization.

We omit verifying the other seven equations.

(c) Assume that G' is an orthodox functor such that $G'H = G$. We claim that $h \cdot\!=\!\cdot\; G'(\rho)$ must satisfy (i), that is

$$G(|\,\rho\,|)^{\iota}\langle \mu_\alpha G'(\rho), \mu_\beta\rangle \cdot\!=\!\cdot\; \mathit{true}\, 0_{G'(\beta)}\,,$$

or more explicitely

$$\epsilon_{G(A\times B)}\langle G(|\,\rho\,|)\, 0_{G'(\beta)}\,, \langle \mu_\alpha G'(\rho), \mu_\beta\rangle\rangle \cdot\!=\!\cdot\; \mathit{true}\, 0_{G'(\beta)}\,.$$

Since $G = G'H$ and $\mu_\alpha \cdot\!=\!\cdot\; G'(\varphi_\alpha)$, this equation in $\mathscr{E}$ may be obtained by applying the orthodox functor G' to the following equation in $T(\mathfrak{A})$:

$$\boldsymbol{\epsilon}_{H(A\times B)} * \boldsymbol{\langle} H(|\,\rho\,|) * \mathbf{0}_\beta\,, \boldsymbol{\langle}\varphi_\alpha * \rho, \varphi_\beta\boldsymbol{\rangle\rangle} \cdot\!=\!\cdot\; \mathbf{true} * \mathbf{0}_\beta\,.$$

To prove the latter equation, we calculate the left-hand side, looking at the definitions of $*$, $\boldsymbol{\epsilon}$, $\boldsymbol{\langle\ \rangle}$, H, and $\mathbf{0}_\beta$. After some tedious details, which we omit, we find that the left-hand side is

$$\begin{aligned} &\{\langle t, y\rangle \in \Omega \times B \mid \exists_{x\in A}\, t = (\langle x, y\rangle \in |\,\rho\,|) \wedge y \in \beta \wedge x \in \alpha \wedge \langle x, y\rangle \in |\,\rho\,|\} \\ &\quad \cdot\!=\!\cdot\; \{\langle t, y\rangle \in \Omega \times B \mid \exists_{x\in A}\, t = \mathit{true} \wedge y \in \beta \wedge x \in \alpha \wedge \langle x, y\rangle \in |\,\rho\,|\} \\ &\quad \cdot\!=\!\cdot\; \{\langle \mathit{true}, y\rangle \in \Omega \times B \mid y \in \beta\}. \end{aligned}$$

On the other hand, the right-hand side is

$$\begin{aligned} &\{\langle t, y\rangle \in \Omega \times B \mid \exists_{v\in 1}\, \langle t, v\rangle \in \mathbf{true} \wedge \langle v, y\rangle \in \mathbf{0}_\beta\} \\ &\quad \cdot\!=\!\cdot\; \{\langle t, y\rangle \in \Omega \times B \mid t = \mathit{true} \wedge \langle 0_1\,, y\rangle \in \mathbf{0}_\beta\}, \end{aligned}$$

which reduces to the same thing.

COROLLARY 11.2. *If $\mathfrak{A}$ is a topos with canonical subobjects, $H\colon \mathfrak{A} \to T(\mathfrak{A})$ is an equivalence of categories.*

Proof. Let $G\colon \mathfrak{A} \to \mathfrak{A}$ be the identity functor. Then the theorem yields $G'\colon T(\mathfrak{A}) \to \mathfrak{A}$ such that $G'H$ is the identity on $\mathfrak{A}$.

On the other hand,

$$\begin{aligned} HG'(\alpha) &= H\,\mathrm{Ker}(\alpha^{\langle}) \\ &\cong \mathrm{Ker}\,H(\alpha^{\langle}) \\ &\cdot\!=\!\cdot\ \alpha; \end{aligned}$$

for H, being an orthodox functor, preserves kernels up to isomorphism, as was pointed out in Section 9, and the final equality was established in the proof of Theorem 11.1.

If we knew that H preserved canonical subobjects, we could replace the isomorphism above by equality, hence the equivalence of categories by an isomorphism. Unfortunately this is not the case. For, if B is a canonical subobject of A in $\mathfrak{A}$, $H(B) \cdot\!=\!\cdot \{y \in B \mid true\}$ is a set of type PB, and this cannot be a canonical subobject of $H(A)$, unless $B = A$.

On the other hand, we can improve Corollary 11.2 by deleting the condition that $\mathfrak{A}$ has canonical subobjects. This will be done in the next section, but the proof will be quite different.

12. Canonical Subobjects May Be Assumed

We recall $T(\mathfrak{A})$, the topos associated with a dogma, and $H\colon \mathfrak{A} \to T(\mathfrak{A})$, the orthodox functor studied in Section 11, which renders Top a reflective subcategory of Dog. We shall answer some obvious questions about H.

Proposition 12.1. *For a dogma $\mathfrak{A}$ the following are equivalent:*

(0) *For each object B of $\mathfrak{A}$, ι_B is mono.*

(1) *For any morphisms $f, g\colon A \to B$ in $\mathfrak{A}$, if $\vdash \forall_{x \in A}\, fx = gx$, then $f \cdot\!=\!\cdot g$. (Internal equality implies external equality.)*

(2) *$H\colon \mathfrak{A} \to T(\mathfrak{A})$ is faithful.*

(3) *There exists a faitful orthodox functor $G\colon \mathfrak{A} \to \mathscr{E}$, where $\mathscr{E}$ is an elementary topos.*

Proof. We shall show (0) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (0).

(0) $\Rightarrow$ (1). Suppose $\vdash \forall_{x \in A}\, fx = gx$, then $\vdash_x fx = gx$, hence $\iota_B fx \cdot\!=\!\cdot_x \iota_B gx$ by Corollary 7.2. By functional completeness, $\iota_B f \cdot\!=\!\cdot \iota_B g$, hence $f \cdot\!=\!\cdot g$ by (0).

(1) $\Rightarrow$ (2). Suppose $H(f) \cdot\!=\!\cdot H(g)$. Then

$$\{\langle y, x\rangle \in B \times A \mid y = fx\} \cdot\!=\!\cdot \{\langle y, x\rangle \in B \times A \mid y = gx\},$$

hence

$$\vdash \forall_{\langle y,x\rangle \in B\times A}\,(y = fx \Leftrightarrow y = gx),$$

from which it easily follows that

$$\vdash \forall_{x\in A}\, fx = gx.$$

Therefore $f \cdot\!=\!\cdot\, g$ by (1).

(2) $\Rightarrow$ (3). This is evident in view of Theorem 11.1.

(3) $\Rightarrow$ (0). From $\iota_B f \cdot\!=\!\cdot\, \iota_B g$ we deduce that $\iota_{G(B)}G(f) \cdot\!=\!\cdot\, \iota_{G(B)}G(g)$, since G is orthodox. Now $\iota_{G(B)}$ is the singleton morphism in a topos, hence mono (see below), and so $G(f) \cdot\!=\!\cdot\, G(g)$. Since G is faithful, $f \cdot\!=\!\cdot\, g$.

Given an object B in a topos, why is the singleton morphism ι_B mono? We shall first establish that our singleton morphism coincides with the usual one.

The diagonal morphism $\langle 1_B, 1_B\rangle: B \to B \times B$ is a (split) mono, hence P-regular; for in a topos every mono is the equalizer of two morphisms into $\Omega \simeq P1$. Its characteristic morphism δ_B may therefore be calculated by Proposition 8.2:

$$\begin{aligned} \ulcorner\delta_B\urcorner &\cdot\!=\!\cdot\, \{z \in B \times B \mid \exists_{y\in B}\, \langle 1_B, 1_B\rangle y = z\} \\ &\cdot\!=\!\cdot\, \{\langle y, y'\rangle \in B \times B \mid y = y'\}. \end{aligned}$$

It has already been pointed out that then $\delta_B^* \cdot\!=\!\cdot\, \iota_B$.

It is of course well-known that the usual singleton morphism is mono. For completeness we include a proof here.

Suppose $f, g: A \to B$ are such that $\delta_B^* f \cdot\!=\!\cdot\, \delta_B^* g$. Passing from $A \to PB$ to $A \times B \to \Omega$, we obtain

$$\delta_B\langle f\pi_{A,B}, \pi'_{A,B}\rangle \cdot\!=\!\cdot\, \delta_B\langle g\pi_{A,B}, \pi'_{A,B}\rangle.$$

Multiplying by $\langle 1, g\rangle$ on the right, we then get

$$\delta_B\langle f, g\rangle \cdot\!=\!\cdot\, \delta_B\langle g, g\rangle \cdot\!=\!\cdot\, \mathit{true}\, 0_B, \qquad \text{hence} \quad f \cdot\!=\!\cdot\, g.$$

We shall say that a dogma is *descriptive* if from $\vdash \forall_{x\in A}\, \exists!_{y\in B}\, p(x, y)$ one may infer that there is a unique $g: A \to B$ such that $\vdash \forall_{x\in A}\, p(x, gx)$, that is, $y = gx \cdot\!=\!\cdot_{\langle x,y\rangle}\, p(x, y)$.

PROPOSITION 12.2. *For any dogma $\mathfrak{A}$, the following conditions are equivalent*:

(0) *For each object B of $\mathfrak{A}$, $\iota_B: B \to PB$ is a P-regular monomorphism.*

(1) *$\mathfrak{A}$ is descriptive.*

(2) *$H: \mathfrak{A} \to T(\mathfrak{A})$ is full and faithful.*

(3) *There exists a full and faithful orthodox functor $G: \mathfrak{A} \to \mathscr{E}$, where $\mathscr{E}$ is an elementary topos.*

Proof. We show (0) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (0).

(0) $\Rightarrow$ (1). This follows from Theorem 8.3.

(1) $\Rightarrow$ (2). Recall that $H(A) \cdot\!=\!\cdot \{x \in A \mid \mathit{true}\}$. Consider any function $\varphi: H(A) \to H(B)$, then

$$\vdash \forall_{x\in A} \, (x \in H(A) \Rightarrow \exists!_{y\in B} \langle y, x\rangle \in |\, \varphi \,|),$$

that is,

$$\vdash \forall_{x\in A} \, \exists!_{y\in B} \langle y, x\rangle \in |\, \varphi \,|.$$

By (1), there is a unique $g: A \to B$ such that

$$y = gx \cdot\!=\!\cdot_{\langle x,y\rangle} \langle y, x\rangle \in |\, \varphi \,|,$$

and from this it easily follows that

$$\{\langle y, x\rangle \in B \times A \mid y = gx\} \cdot\!=\!\cdot \, |\, \varphi \,|,$$

that is, $|\, H(g)| \cdot\!=\!\cdot |\, \varphi \,|$, hence $H(g) \cdot\!=\!\cdot \varphi$.

(2) $\Rightarrow$ (3). This is evident.

(3) $\Rightarrow$ (0). Consider the following diagram in $\mathcal{A}$:

$$B \xrightarrow{\iota_B} PB \underset{g}{\overset{f}{\rightrightarrows}} \Omega$$

where

$$\ulcorner f \urcorner \cdot\!=\!\cdot \{v \in PB \mid \exists_{y\in B} \, \iota_B y = v\}$$

and

$$g \cdot\!=\!\cdot \mathit{true} \, 0_{PB} \, .$$

Applying the orthodox functor G to this, we obtain a similar diagram in $\mathscr{E}$. Moreover,

$$\ulcorner G(f) \urcorner \cdot\!=\!\cdot \{v \in PG(B) \mid \exists_{y\in G(B)} \, G(\iota_B) y = v\},$$

in view of Corollary 5.4. But then we know from Proposition 8.2 that $G(\iota_B)$ is an equalizer of $G(f)$ and $G(g)$. If G is full and faitful, ι_B is already an equalizer of f and g.

The following result strengthens Corollary 11.2 and justifies our definition of the category Top.

PROPOSITION 12.3. *Every topos $\mathcal{A}$ is equivalent to a topos with canonical subobjects; in fact, $H: \mathcal{A} \to T(\mathcal{A})$ is an equivalence of categories.*

Proof. We know from the last two propositions that $H: \mathcal{O}\!\mathcal{l} \to T(\mathcal{O}\!\mathcal{l})$ is full and faithful. We now claim that every object of $T(\mathcal{O}\!\mathcal{l})$ is isomorphic to some object in the image of H.

Suppose $\alpha: 1 \to PA$ is any object of $T(\mathcal{O}\!\mathcal{l})$. Let $m: B \to A$ be an equalizer of $\alpha^{(}$ and $true\, 0_A$. We shall find an isomorphism $\rho: \alpha \to H(B)$. Take

$$| \rho | \cdot\!=\!\cdot \{\langle y, x\rangle \in B \times A \mid x = my \wedge x \in \alpha\}.$$

Clearly, ρ^{-1} is a function, but why is ρ? We want to show that

$$\vdash \forall_{x\in A}\, x \in \alpha \Rightarrow \exists !_{y\in B}\, x = my,$$

that is,

$$x \in \alpha \vdash_x \exists !_{y\in B}\, x = my.$$

Now $\alpha^{(}$ is the characteristic morphism of m, hence, by Proposition 8.2,

$$\alpha \cdot\!=\!\cdot \{x \in A \mid \exists_{y\in A}\, my = x\},$$

and, therefore,

$$x \in \alpha \vdash_x \exists_{y\in B}\, my = x.$$

The uniqueness follows from the following known lemma, a proof of which we include for completeness.

LEMMA 12.4. *In a topos, $m: B \to A$ is a monomorphism if and only if*

$$\vdash \forall_{\langle y,y'\rangle\in B\times B}\, (my = my' \Rightarrow y = y').$$

Proof. Suppose the condition holds. Let $f, g: C \to B$ and take an indeterminate z of type C. By universal specification, we obtain

$$\vdash_z mfz = mgz \Rightarrow fz = gz.$$

Suppose $mf \cdot\!=\!\cdot mg$, then

$$\vdash \forall_{z\in C}\, mfz = mgz,$$

by Proposition 7.1, hence

$$\vdash \forall_{z\in C}\, fz = gz,$$

and so $f \cdot\!=\!\cdot g$, by Proposition 12.1. Thus m is a monomorphism.

Conversely, suppose m is mono. Let $\langle f, g\rangle: C \to B \times B$ be an equalizer of h and $true\, 0_{B\times B}$ where

$$\ulcorner h \urcorner \cdot\!=\!\cdot \{\langle y, y'\rangle \in B \times B \mid my = my'\}.$$

Then, for an indeterminate z of type C, we have

$$h\langle fz, gz\rangle \cdot\!=\!\cdot_z \mathit{true},$$

that is,

$$mfz = mgz \cdot\!=\!\cdot_z \mathit{true},$$

so that $mf \cdot\!=\!\cdot mg$, by Proposition 12.1. Since m is mono, it follows that $f \cdot\!=\!\cdot g$. Therefore, h is the unique characteristic morphism of $\langle f, f\rangle$, and so, by Proposition 8.2,

$$\ulcorner h \urcorner \cdot\!=\!\cdot \{\langle y, y'\rangle \in B \times B \mid \exists_{z\in C} \langle y, y'\rangle = \langle f, f\rangle z\}.$$

Thus,

$$\langle y, y'\rangle \in \ulcorner h \urcorner \vdash_{\langle y,y'\rangle} \exists_{z\in C} \langle y, y'\rangle = \langle fz, fz\rangle,$$

and therefore

$$my = my' \vdash_{\langle y,y'\rangle} y = y',$$

from which the condition follows easily.

13. Additional Equational Structure

In this section we consider predogmas and dogmas with additional equational structure.

We recall that a *Cartesian closed* category is a Cartesian category in which every object B admits all powers B^A with morphisms $\epsilon_{B,A}$: $B^A \times A \to B$ such that $\mathrm{Hom}(C, B^A) \cong \mathrm{Hom}(C \times A, B)$, whereas in a predogma this was only so for $B = \Omega$. Functional completeness holds for Cartesian closed categories (see [16]), but it permits an interesting alternative formulation: Given an indeterminate x: $1 \to A$ and a polynomial $f(x)$: $C \to B$, there exists a unique h: $C \to B^A$ such that $f(x) \cdot\!=\!\cdot_x \epsilon_{B,A} \langle h, x0_C\rangle$. We write $h \cdot\!=\!\cdot \lambda_x f(x)$, and this agrees with the λ-conversion of Church in the special case when $C = 1$.

In the topos $T(\mathfrak{A})$ generated by a dogma $\mathfrak{A}$ one may define β^α for sets β: $1 \to PB$ and α: $1 \to PA$ by

$$\beta^\alpha \cdot\!=\!\cdot \{w \in P(B \times A) \mid w \subseteq \beta \times \alpha \wedge \forall_{x\in A} (x \in \alpha \Rightarrow \exists!_{y\in B}\, y \in \beta \wedge \langle y, x\rangle \in w\}.$$

In this way $T(\mathfrak{A})$ is easily shown to be Cartesian closed, and so one recaptures the result of Kock [14] that every topos (as defined here) is Cartesian closed.

Why did we not require that a dogma is Cartesian closed? Because the canonical functor H: $\mathfrak{A} \to T(\mathfrak{A})$ constructed in Section 11 does not preserve internal powers, that is, in general $H(B^A) \neq H(B)^{H(A)}$. This is also the reason why we wrote PA instead of Ω^A.

In [16], Theorem 4.2 asserts that functional completeness holds for Cartesian closed categories with additional equational structure, consisting of morphisms $f: 1 \to C$, equations $\varphi(f) = \psi(f)$ for all $f: 1 \to C$, and implications between such equations. Unfortunately the short proof of the theorem (on page 287 of the cited paper) contains an error, which was discovered by Obtutowicz. Presumably, the theorem is wrong in the sense in which it was intended, which required the equations to remain valid if f is replaced by a polynomial $f(x)$. The theorem holds trivially if additional equational structure consists only of morphisms $f: 1 \to C$ and equations of the form $\varphi \cdot\!=\!\cdot\, \psi$. The difficulty with the theorem as stated was how to deduce from the fact that $\varphi(f) \cdot\!=\!\cdot\, \psi(f)$ for all $f: 1 \to C$ that $\kappa_z\varphi(z) \cdot\!=\!\cdot\, \kappa_z\psi(z)$. In some examples this may be possible.

Thus, in Example 4.6 of the cited paper, the additional structure consisted of finite coproducts. This structure may be presented by morphisms

$$0 \to C, \qquad A \to A + B, \qquad B \to A + B, \qquad C^A \times C^B \to C^{A+B}$$

and equations to ensure that $C^0 \cong 1$ and $C^{A+B} \cong C^A \times C^B$. Example 4.8 may also be vindicated, since the adjunction between $\forall_A$ and $\pi^*_{2,A}$ can be expressed by equations as in Section 4 above.

Functional completeness clearly remains valid for dogmas (or predogmas) with additional equational structure consisting of morphisms $1 \to C$ and equations $\varphi \cdot\!=\!\cdot\, \psi$. What additional structure could reasonably be demanded of a dogma? One would like to assure that the topos generated by it should have some of the additional structure one requires for the category of sets. The existence of finite colimits in a topos is assured by a result of Mikkelsen. There remain the following postulates a topos should satisfy to conform to Lawvere's elementary theory of the category of sets: the axiom of Booleanness ($1 + 1 \cong \Omega$), the axiom of choice, the axiom of infinity, and the axiom which says that $\mathrm{Hom}(1, \Omega)$ has exactly two elements.

The axiom of infinity for a topos was originally stated in the form of the Peano–Lawvere axiom, but Freyd [9] showed that this is equivalent to the existence of an object N (not necessarily the natural number object) and an isomorphism $N + 1 \cong N$. The axiom of infinity and the axiom of Booleanness thus both have the form $A + B \cong C$, and we note that $\mathrm{Hom}(1, C)$ is nonempty in both cases. What equational structure in a dogma corresponds to this? The answer is given by Proposition 13.1 below, but first a definition.

We call an object A of a dogma *empty* if $\mathrm{Hom}(1, A)$ is empty, we call it *P-regular* if $\iota_A: A \to PA$ is a P-regular monomorphism.

PROPOSITION 13.1. *Suppose $\mathfrak{A}$ is a dogma in which all nonempty objects are P-regular. If A and B are nonempty and $\kappa: A \to C$, $\lambda: B \to C$ induce an isomorphism $PC \cong PA \times PB$, then (C, κ, λ) is a coproduct of A and B.*

Proof. We want to show that the mapping $\mathrm{Hom}(C, D) \to \mathrm{Hom}(A, D) \times \mathrm{Hom}(B, D)$ which sends $h: C \to D$ onto $(h\kappa, h\lambda)$ is one-to-one and onto. When D is empty this is clear; otherwise $\iota_D: D \to PD$ is an equalizer of $u, v: PD \to PE$, say. Given $f: A \to D$, $g: B \to D$, then $\langle \iota_D f, \iota_D g\rangle$ is in $\mathrm{Hom}(A, PD) \times \mathrm{Hom}(B, PD)$. Let ψ_D be the one-to-one correspondence of this with $\mathrm{Hom}(C, PD)$ obtained by tracing

$$\begin{aligned}\mathrm{Hom}(A, PD) \times \mathrm{Hom}(B, PD) &\cong \mathrm{Hom}(D, PA) \times \mathrm{Hom}(D, PB)\\ &\cong \mathrm{Hom}(D, PA \times PB) \cong \mathrm{Hom}(D, PC)\\ &\cong \mathrm{Hom}(C, PD).\end{aligned}$$

Then we have to show that $h' \cdot\!=\!\cdot\, \psi_D\langle \iota_D f, \iota_D g\rangle$ factors through D, and this is done by verifying that $uh' \cdot\!=\!\cdot\, vh'$. We shall skip the details. [First show that ψ_D is the inverse of $\langle \mathrm{Hom}(\kappa, PD), \mathrm{Hom}(\lambda, PD)\rangle$. Then get $uh'\kappa \cdot\!=\!\cdot\, vh'\kappa$ and $uh'\lambda \cdot\!=\!\cdot\, vh'\lambda$ and finally $\iota_{PE} uh' \cdot\!=\!\cdot\, \iota_{PE} vh'$.]

Instead of postulating a morphism $PA \times PB \to PC$ and equations which it should satisfy, we may equally well postulate

$$\vdash \forall_{\langle x,y\rangle \in PA \times PB}\, \exists!_{z \in PC}\, \langle P\kappa, P\lambda\rangle z = \langle x, y\rangle.$$

Here P is taken to be a contravariant functor. Indeed, this condition is surely necessary if $\langle P\kappa, P\lambda\rangle$ is to have an inverse. That it is also sufficient follows easily from Theorem 8.3, in view of the observation that ι_{PC} is a P-regular monomorphism; in fact, it splits.

Proposition 13.1 leads to a formulation of the Boolean axiom which says that truth functions are determined by their values at *true* and *false*. Not unexpectedly, this is equivalent to

$$\vdash \forall_{t \in \Omega}\, \neg\neg\, t \Rightarrow t, \tag{10}$$

where, of course, $\neg t$ is short for $t \Rightarrow$ *false*.

Proposition 13.1 leads to a formulation of the axiom of infinity which is a little bit stronger than the conjunction of the following two conditions:

$$\vdash \forall_{x \in N}\, \neg\, (sx = 0), \tag{11a}$$

$$\vdash \forall_{\langle x,y\rangle \in N \times N}\, (sx = sy \Rightarrow x = y). \tag{11b}$$

(It also says that every element of N is either 0 or a successor.) In a dogma, these two conditions already allow us to define the *set of natural numbers* (a set, not a type) as the intersection of all those subsets z of N such that $0 \in z$ and $z \subseteq s^{-1}z$. In a topos, sets and types are the same thing, that is, objects of the topos, and we may assume without loss in generality that N is already this intersection. Thus we have the classical *axiom of induction*:

$$\vdash \forall_{z\in PN}\,((0 \in z \wedge z \subseteq s^{-1}z) \Rightarrow \forall_{x\in N}\, x \in z), \tag{11c}$$

where

$$s^{-1}z \;\cdot\!=\!\cdot_z\; \{x \in N \mid sx \in z\}.$$

Not surprisingly, (11a) to (11c) imply the Peano–Lawvere axiom, as may be shown. We summarize the results of our discussion:

PROPOSITION 13.2. *A topos is Boolean if and only if it satisfies* (10). *It has a natural number object if and only if it has an object* $(N, 0, s)$ *satisfying* (11a) *and* (11b). $(N, 0, s)$ *is a natural number object if and only if also* (11c).

In order to discuss the axiom of choice for dogmas, we shall say that an object A of the dogma is equipped with a *choice morphism* $f\colon PA \to A$ provided

$$\vdash \forall_{u\in PA}\,(\exists_{x\in A}\, x \in u \Rightarrow fu \in u).$$

Of course, this implies that $\mathrm{Hom}(1, A)$ is nonempty, since $\mathrm{Hom}(1, PA)$ is nonempty.

PROPOSITION 13.3. *For a Boolean dogma $\mathfrak{A}$ the following two statements are equivalent*:

(1) *Every nonempty P-regular object has a choice morphism.*

(2) *Epimorphisms split in $T(\mathfrak{A})$.*

Proof. Assume (1) and let $\varphi\colon \beta \to \alpha$ be an epimorphism in $T(\mathfrak{A})$, where $\alpha\colon 1 \to FA$ and $\beta\colon 1 \to PB$. Without loss in generality we may assume that B is nonempty and P-regular; for β is isomorphic to $\beta'\colon 1 \to PPB$, where

$$\begin{aligned} \beta' \;&\cdot\!=\!\cdot\; \{\{y\} \in PB \mid y \in \beta\} \\ &\cdot\!=\!\cdot\; \{v \in PB \mid \exists_{y\in B}\,(y \in \beta \wedge v = \{y\})\}. \end{aligned}$$

By Lemma 13.4 below, we have

$$\vdash \forall_{x\in A}\,(x \in \alpha \Rightarrow \exists_{y\in B}\, \langle x, y\rangle \in |\,\varphi\,|),$$

that is,

$$\vdash_x \exists_{y\in B}\, y \in \theta(x),$$

where

$$\theta(x) \;\cdot\!=\!\cdot_x\; \{y \in B \mid x \in \alpha \Rightarrow \langle x, y\rangle \in |\,\varphi\,|\}.$$

Now let $f: PB \to B$ be the choice morphism for B, then

$$\vdash_x f\theta(x) \in \theta(x),$$

that is,

$$\vdash_x x \in \alpha \Rightarrow \langle x, f\theta(x)\rangle \in |\,\varphi\,|.$$

Define $\psi: \alpha \to \beta$ by

$$|\,\psi\,| \cdot\!=\!\cdot \{\langle y, x\rangle \in B \times A \mid x \in \alpha \wedge y = f\theta(x)\},$$

then it is easily verified that $\varphi * \psi \cdot\!=\!\cdot 1_\alpha$, and so (1) implies (2).

Conversely, assume (2), and suppose that A is nonempty and P-regular. Consider the morphisms $\varphi: \beta \to \alpha$ in $T(\mathcal{O})$, where

$$\alpha \cdot\!=\!\cdot \{u \in PA \mid \exists_{x\in A}\, x \in u\},$$
$$\beta \cdot\!=\!\cdot \{\langle u, x\rangle \in PA \times A \mid x \in u\}.$$
$$|\,\varphi\,| \cdot\!=\!\cdot \{\langle v, \langle u, x\rangle\rangle \in PA \times (PA \times A) \mid x \in u \wedge u = v\}.$$

In view of Lemma 13.4, it is easily seen that φ is an epimorphism. Therefore, there exists $\psi: \alpha \to \beta$ such that $\varphi * \psi \cdot\!=\!\cdot 1_\alpha$, that is,

$$\vdash_{\langle u,v\rangle} \exists_{x\in A}\,(x \in u \wedge \langle\langle u, x\rangle, v\rangle \in |\,\psi\,|) \Leftrightarrow u = v.$$

We may extend ψ to $\chi: \{u \in PA \mid \mathit{true}\} \to \alpha$ by arbitrarily setting $\chi u = \langle u, a\rangle$, where a is a given element of $\mathrm{Hom}(1, A)$, in case $\neg\,\exists_{x\in A}\, x \in u$. The formal definition of χ is

$$|\,\chi\,| \cdot\!=\!\cdot \{\langle\langle u, x\rangle, u\rangle \mid \langle\langle u, x\rangle, u\rangle \in |\,\psi\,| \vee (\neg\,\exists_{y\in A}\, y \in u \wedge x = a)\}.$$

It is easily seen that

$$\vdash \forall_{u\in PA}\, \exists!_{x\in A}\, \langle\langle u, x\rangle, u\rangle \in |\,\chi\,|,$$

hence it follows from Proposition 8.3 that there is a unique $f: PA \to A$ such that

$$\vdash \forall_{u\in PA}\, \langle\langle u, fu\rangle, u\rangle \in |\,\chi\,|,$$

and one easily checks that f is a choice morphism for A. Thus (2) implies (1).

It remains to prove the following lemma, which is well-known.

LEMMA 13.4. *In a topos $e: B \to A$ is an epimorphism if and only if*

$$\vdash \forall_{x\in A}\, \exists_{y\in B}\, x = ey.$$

Proof. Suppose the condition is satisfied and $f, g: A \to C$ are such that $fe \cdot\!=\!\cdot\, ge$. Then we easily obtain

$$\vdash \forall_{x \in A}\, fx = gx,$$

hence $f \cdot\!=\!\cdot\, g$, by Proposition 12.1 and so e is epi.

Conversely, suppose e is an epimorphism. By functional completeness, there exists a unique morphism $f: A \to \Omega$ such that

$$fx \cdot\!=\!\cdot_x \exists_{y \in B}\, x = ey,$$

and it follows that

$$fey \cdot\!=\!\cdot_y\, true,$$

hence $fe \cdot\!=\!\cdot\, true\, 0_B \cdot\!=\!\cdot\, true\, 0_A e$, by functional completeness. Therefore, $f \cdot\!=\!\cdot\, true\, 0_A$, and the condition follows.

COROLLARY 13.5. *For a Boolean topos (more generally, for a Boolean dogma in which all objects are P-regular) the following statements are equivalent:*

(1) *Every nonempty object has a choice morphism.*

(2) *Epimorphisms split.*

Actually, (1) can be reformulated without the condition that the object be nonempty:

For every object A there exists a morphism $g: PA \times A \to A$ such that

$$\vdash \forall_{\langle u,x\rangle \in PA \times A}\, (\exists_{y \in A}\, y \in u \Rightarrow g\langle u, x\rangle \in u).$$

The final condition which the topos of sets should satisfy is that Hom$(1, \Omega)$ has two elements. It is not difficult to see that this statement for the dogma $\mathcal{O}\!\mathcal{l}$ assures that Hom$(\mathbf{1}, \mathbf{\Omega})$ has two elements in $T(\mathcal{O}\!\mathcal{l})$. The reason for this is that each morphism $\mathbf{1} \to \mathbf{\Omega}$ in $T(\mathcal{O}\!\mathcal{l})$ has the form $H(f)$, where $f: 1 \to \Omega$ in $\mathcal{O}\!\mathcal{l}$, in view of Theorem 8.3 and the observation that $\Omega \to P\Omega$ is P-regular (in fact it splits). Unfortunately, the condition that Hom$(1, \Omega)$ has two elements is not equational.

14. FREE TOPOSES

One may form the free dogma $D(\mathcal{X})$ generated by a category (or graph) $\mathcal{X}$ using the methods of "deductive systems" as in [15]. As objects of $D(\mathcal{X})$ one takes all "formulas" made up from objects of $\mathcal{X}$, 1, and Ω with the help of the operations $\times$ and P. As morphisms in $D(\mathcal{X})$ one takes all proofs of "sequents" $A \to B$, regarding $D(\mathscr{X})$ as a deductive system in which all morphisms $X \to Y$ of $\mathscr{X}$ appear as postulates. Finally one imposes all those equations between proofs which are needed to ensure that $D(\mathscr{X})$ is a dogma and that $\mathscr{X} \to D(\mathscr{X})$ is a functor. (The last proviso is not necessary if $\mathscr{X}$ is only a graph.)

It then follows that $T(D(\mathscr{X}))$ is the free topos with canonical subobjects generated by $\mathscr{X}$.

In the same way one may form the free Boolean topos or the free topos with axiom of infinity or the free Boolean topos with axiom of infinity by first forming the free dogma satisfying whichever of Eqs. (10) or (11) are required. If one wishes to nail down N as the natural number object, one should impose (11c).

Suppose we want to form the free topos with axiom of choice generated by the empty category. We first form the free Boolean dogma generated by the empty category and note that each object in it is regular and nonempty, in view of Lemma 14.1 below. Then we postulate a choice morphism for each object and form the topos generated by this dogma.

LEMMA 14.1. *In the free dogma generated by the empty category, all morphisms $\iota_A: A \to PA$ split, hence all objects are P-regular and nonempty. Thus H is full and faithful for this dogma.*

Proof. We claim that for each object A there exists a morphism $\sigma_A: PA \to A$ such that $\sigma_A \iota_A \cdot\!=\!\cdot\, 1_A$. We construct σ_A as follows:

$$\begin{aligned}
\sigma_1 &\cdot\!=\!\cdot\, 0_{P1}, \\
\sigma_\Omega u &\cdot\!=\!\cdot_u\, \mathit{true} \in u, \\
\sigma_{PA} v &\cdot\!=\!\cdot_v\, \{x \in A \mid \exists_{y \in PA}\, y \in v \wedge x \in y\}, \\
\sigma_{A \times B} w &\cdot\!=\!\cdot_w\, \langle \sigma_A\{x \in A \mid \exists_{y \in B} \langle x, y\rangle \in w\}, \sigma_B\{y \in B \mid \exists_{x \in A} \langle x, y\rangle \in w\}\rangle.
\end{aligned}$$

Here u, v, and w are indeterminates of types $P\Omega$, $P(PA)$, and $P(A \times B)$, respectively.

Suppose we want to form the free topos with axiom of infinity and axiom of choice generated by the empty category. We can apply the same argument as above, provided we know that the morphism $\iota_N: N \to PN$ splits. This can be proved easily with the help of Theorem 8.3, if it is known that N is P-regular. I do not know whether this can be proved in general. No matter, we can postulate a morphism $\sigma_N: PN \to N$ such that $\sigma_N \iota_N \cdot\!=\!\cdot\, 1_N$ and then proceed as above.

In the above dogma, $p \cdot\!=\!\cdot\, q$ surely means that the equivalence of p and q is provable according to the usual rules of formal number theory, hence $\mathrm{Hom}(1, \Omega)$ is the Lindenbaum algebra for this theory. Now Gödel's incompleteness theorem asserts that there are undecidable propositions, hence $\mathrm{Hom}(1, \Omega)$ has more than two elements. Can we form the free topos with axiom of infinity and axiom of choice and such that $\mathrm{Hom}(1, \Omega)$ has two elements generated by the empty category? If this were possible, there would be a natural candidate for the category of sets, and this is probably too much to expect.

While it may be difficult to produce a natural candidate for the category of sets which will satisfy classical mathematicians, the free topos with natural number object generated by the empty category looks very much like an intuitionist's idea of a category of sets.

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